Illustrative Mathematics

N-RN Rational or Irrational?

Alignments to Content Standards

- Alignment: N-RN.A.2
- Alignment: N-RN.B

Tags

• *This task is not yet tagged.*

In each of the following problems, a number is given. If possible, determine whether the given number is rational or irrational. In some cases, it may be impossible to determine whether the given number is rational or irrational. Justify your answers.

a. $4+\sqrt{7}$

b. $\dfrac{\sqrt{45}}{\sqrt{5}}$

c. $\dfrac{6}{\pi}$

d. $\sqrt{2}+\sqrt{3}$

e. $\dfrac{2+\sqrt{7}}{2a+\sqrt{7a^2}}$, where a is a positive integer

f. $x+y$, where x and y are irrational numbers

Commentary

This task makes for a good follow-up task on rational irrational numbers after that the students have been acquainted with some of the more basic properties (e.g., that π and square roots non-square integers are irrational, and that a rational plus an irrational is again irrational, etc.), asking students to reason about rational and irrational numbers (N-RN.3) in a variety of ways. In addition to eliciting several different types of reasoning, the task requires students to rewrite radical expressions in which the radicand is divisible by a perfect square (N-RN.2).

The solutions to this task are written as formal arguments; teachers are encouraged to engage students in a dialogue (or have them engage each other in groups) to help them develop rigorous arguments for the rationality and irrationality of each of the given numbers.

Solutions

Solution: Solution

a. We know that $\sqrt{7}$ is irrational, so we conjecture that $4+\sqrt{7}$ is irrational as well. To prove this, suppose that $4+\sqrt{7}$ were a rational number $\frac{a}{b}$, where a and b are integers. Then we would have

$$4+\sqrt{7}=\frac{a}{b} \qquad \Rightarrow \qquad \sqrt{7}=\frac{a}{b}-4.$$

But then $\sqrt{7}$ would be a difference of two rational numbers, which can be seen to be rational:

$$\sqrt{7}=\frac{a-4b}{b}$$

Since $a-4b$ and b are integers, this would mean that $\sqrt{7}$ is rational, which we know to be false. So $4+\sqrt{7}$ must be irrational. Note that we may use a similar argument to show that the sum of any rational number and any irrational number is irrational.

b. We know that $\sqrt{45}=\sqrt{5\cdot 9}=\sqrt{5}\cdot\sqrt{9}=3\sqrt{5}$. So

$$\frac{\sqrt{45}}{\sqrt{5}}=\frac{3\sqrt{5}}{\sqrt{5}}=3,$$

which is rational.

c. We conjecture that $\frac{6}{\pi}$ is irrational. To prove this, suppose that $\frac{6}{\pi}$ were a rational number $\frac{a}{b}$, where a and b are integers. Then we would have

$$\frac{6}{\pi}=\frac{a}{b} \qquad \Rightarrow \qquad \frac{6b}{a}=\pi.$$

Since $6b$ and a are integers, this means that π is a rational number, which we know to be false. Therefore, $\frac{6}{\pi}$ cannot be a rational number. In fact, we may use a similar argument to show that if r is any nonzero rational number and x is any irrational number, then rx and $\frac{r}{x}$ are irrational numbers.

d. We conjecture that $\sqrt{2}+\sqrt{3}$ is irrational. If it were rational, then its square $(\sqrt{2}+\sqrt{3})^2$ would also be rational. But we have

$$(\sqrt{2}+\sqrt{3})^2 = 2+2\cdot\sqrt{2}\cdot\sqrt{3}+3 = 5+2\sqrt{6}.$$

We know that $\sqrt{6}$ is irrational, and thus $5+2\sqrt{6}$ is also irrational (since doubling an irrational number produces an irrational number, as does adding 5 to an irrational number). Since $(\sqrt{2}+\sqrt{3})^2$ is irrational, $\sqrt{2}+\sqrt{3}$ must be irrational as well.

e. We can rewrite the given expression:

$$\begin{aligned}\frac{2+\sqrt{7}}{2a+\sqrt{7a^2}} &= \frac{2+\sqrt{7}}{2a+\sqrt{7}\cdot\sqrt{a^2}}\\ &= \frac{2+\sqrt{7}}{2a+\sqrt{7}\cdot a}\\ &= \frac{2+\sqrt{7}}{a(2+\sqrt{7})}\\ &= \frac{1}{a}.\end{aligned}$$

Since a is a positive integer, this number is rational. (Note that rewriting $\sqrt{a^2}=a$ requires knowing that $a>0$. In general, $\sqrt{a^2}=|a|$.)

f. The given number may be irrational; part (c) gives an example of a situation in which the sum of two irrational numbers is irrational. However, $x+y$ could be a rational number. Suppose that $x=\pi$ and $y=-\pi$. We know that x is irrational, and y is also irrational since the opposite of an irrational number is irrational. But $x+y$ is zero, which is clearly rational.

Therefore, the sum of two irrational numbers can be rational or irrational.