Secondary One Mathematics: An Integrated Approach

Module 1
Systems of Equations and Inequalities

By

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In partnership with the
Utah State Office of Education

Module 1 – Systems of Equations and Inequalities

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Pet Sitters
A Develop Understanding Task

The Martinez twins, Carlos and Clarita, are trying to find a way to make money during summer vacation. When they overhear their aunt complaining about how difficult it is to find someone to care for her pets while she will be away on a trip, Carlos and Clarita know they have found the perfect solution. Not only do they have a large, unused storage shed on their property where they can house animals, they also have a spacious fenced backyard where the pets can play.

Carlos and Clarita are making a list of some of the issues they need to consider as part of their business plan to care for pet cats and dogs while their owners are on vacation.

- *Space*: Cat pens will require 6 ft^2 of space, while dog runs require 24 ft^2. Carlos and Clarita have up to 360 ft^2 available in the storage shed for pens and runs, while still leaving enough room to move around the cages.

- *Start-up Costs*: Carlos and Clarita plan to invest much of the $1280 they earned from their last business venture to purchase cat pens and dog runs. It will cost $32 for each cat pen and $80 for each dog run.

Of course, Carlos and Clarita want to make as much money as possible from their business, so they are trying to determine how many of each type of pet they should plan to accommodate. They plan to charge $8 per day for boarding each cat and $20 per day for each dog.

After surveying the community regarding the pet boarding needs, Carlos and Clarita are confident that they can keep all of their boarding spaces filled for the summer.

So the question is, how many of each type of pet should they prepare for? Their dad has suggested the same number of each, perhaps 12 cats and 12 dogs. Carlos thinks they should plan for more dogs, since they can charge more. Clarita thinks they should plan for more cats since they take less space and time, and therefore they can board more.

What do you think? What recommendations would you give to Carlos and Clarita, and what argument would you use to convince them that your recommendation is reasonable?

Ready, Set, Go!

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Ready

Determine which ordered pair satisfies the system of linear equations, then graph both equations and show the point of intersection to the right of the problem. Be sure to label axes and provide a scale.

1. $y = 3x - 2$
   $y = -x$

   a. (1, 4)

   b. (2, 9)

   c. $\left(\frac{1}{2}, \frac{-1}{2}\right)$

2. $y = 2x - 3$
   $y = x + 5$

   a. (8, 13)

   b. (-7, 6)

   c. (0, 4)

Solve the following systems by graphing. Check the solution by evaluating both equations at the point of intersection.

3. $y = x + 3 \text{ and } y = -2x + 3$

4. $y = 3x - 6 \text{ and } y = -x$

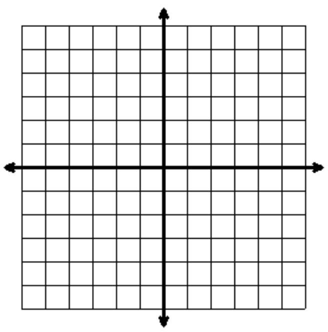
Set

5. A theater wants to take in at least \$2000 for a certain matinee. Children's tickets cost \$5 each and adult tickets cost \$10 each. Find five combinations of children and adult tickets that will make their goal.

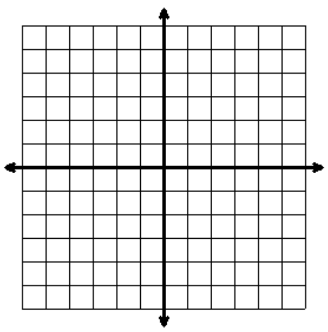
Go

Graph each equation below, then determine if the point (3,5) is a solution to the equation. Find two other points that are solutions to the equation and show these points on the graph.

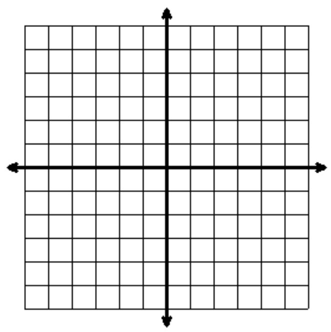
6. $y = 2x-1$



7. $y = \frac{1}{3}x + 2$



8. $y = -3x + 5$



Need help? Check out these related videos:

https://www.youtube.com/watch?v=vo-CXaCf1I4

Too Big or Not Too Big, That is the Question

A Solidify Understanding Task

As Carlos is considering the amount of money available for purchasing cat pens and dog runs (see below) he realizes that his father's suggestion of boarding "the same number of each, perhaps 12 cats and 12 dogs" is too big. Why?

- *Start-up Costs*: Carlos and Clarita plan to invest much of the $1280 they earned from their last business venture to purchase cat pens and dog runs. It will cost $32 for each cat pen and $80 for each dog run.

1. Find at least 5 more combinations of cats and dogs that would be "too big" based on this *Start-up Cost* **constraint.** Plot each of these combinations as points on a coordinate grid using the same color for each point.

2. Find at least 5 combinations of cats and dogs that would not be "too big" based on this *Start-up Cost* constraint. Plot each of these combinations as points on a coordinate grid using a different color for the points than you used in #1.

3. Find at least 5 combinations of cats and dogs that would be "just right" based on this *Start-up Cost* constraint. That is, find combinations of cat pens and dog runs that would cost exactly $1280. Plot each of these combinations as points on a coordinate grid using a third color.

4. What do you notice about these three different collections of points?

5. Write an equation for the line that passes through the points representing combinations of cat pens and dog runs that cost exactly $1280. What does the slope of this line represent?

Carlos and Clarita don't have to spend all of their money on cat pens and dog runs, unless it will help them maximize their profit.

6. Shade all of the points on your coordinate grid that **satisfy** the *Start-up Costs* constraint.

7. Write a mathematical rule to represent the points shaded in #6. That is, write an inequality whose **solution set** is the collection of points that satisfy the *Start-up Costs* constraint.

In addition to *start-up costs,* Carlos needs to consider how much space he has available, base on the following:

- *Space*: Cat pens will require 6 ft^2 of space, while dog runs require 24 ft^2. Carlos and Clarita have up to 360 ft^2 available in the storage shed for pens and runs, while still leaving enough room to move around the cages.

8. Write an inequality to represent the solution set for the *space* constraint. Shade the solution set for this inequality on a different coordinate grid.

Too Big or Not Too Big, That is the Question, part 1

A Solidify Understanding Task

As Carlos is considering the amount of money available for purchasing cat pens and dog runs (see below) he realizes that his father's suggestion of boarding "the same number of each, perhaps 12 cats and 12 dogs" is too big. Why?

- *Start-up Costs*: Carlos and Clarita plan to invest much of the $1280 they earned from their last business venture to purchase cat pens and dog runs. It will cost $32 for each cat pen and $80 for each dog run.

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2. Find at least 5 combinations of cats and dogs that would not be "too big" based on this *Start-up Cost* constraint. Plot each of these combinations as points on a coordinate grid using a different color for the points than you used in #1.

3. Find at least 5 combinations of cats and dogs that would be "just right" based on this *Start-up Cost* constraint. That is, find combinations of cat pens and dog runs that would cost exactly $1280. Plot each of these combinations as points on a coordinate grid using a third color.

4. What do you notice about these three different collections of points?

5. Write an equation for the line that passes through the points representing combinations of cat pens and dog runs that cost exactly $1280. What does the slope of this line represent?

Too Big or Not Too Big, That is the Question, part 2

A Solidify Understanding Task

Carlos and Clarita don't have to spend all of their money on cat pens and dog runs, unless it will help them maximize their profit.

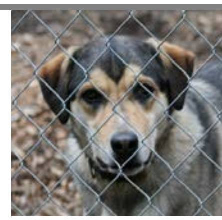
1. Shade all of the points on your coordinate grid that **satisfy** the *Start-up Costs* constraint.

2. Write a mathematical rule to represent the points shaded in #1. That is, write an inequality whose **solution set** is the collection of points that satisfy the *Start-up Costs* constraint.

In addition to *start-up costs,* Carlos needs to consider how much space he has available, base on the following:

- *Space*: Cat pens will require 6 ft^2 of space, while dog runs require 24 ft^2. Carlos and Clarita have up to 360 ft^2 available in the storage shed for pens and runs, while still leaving enough room to move around the cages.

3. Write an inequality to represent the solution set for the *space* constraint. Shade the solution set for this inequality on a different coordinate grid.



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Ready, Set, Go!

Ready

Determine whether each of the given points are solutions to the following linear equations.

1. $3x + 2y = 12$
   a. (2, 4)
   b. (3, 2)
   c. (4, 0)
   d. (0, 6)

2. $5x – y = 10$
   a. (2, 0)
   b. (3, 0)
   c. (0, -10)
   d. (1, 1)

Find the value that will make each ordered pair a solution to the given equations.

3. $x + y = 6$
   a. (2, __)
   b. (0, __)
   c. (__, 0)

4. $2x + 4y = 8$
   a. (2, __)
   b. (0, __)
   c. (__, 0)

5. $3x – y = 8$
   a. (2, __)
   b. (0, __)
   c. (__, 0)

Set

Graph the following inequalities on the coordinate plane. Plot points to make sure the correct region is shaded.

6. $y \leq 4x + 3$

7. $x < 20$

8. $y > -\frac{x}{2} - 6$

9. $y \geq -5$

Go

Follow the directions for each problem below.

10. $14 - 2x < 20$

a. Solve for x.

b. Draw a number line below, and show where the solution set to this problem is.

c. Pick an $x-$value which, according to your drawing, is *inside* the solution set. Plug it into the original inequality $14 - 2x < 20$. Does the inequality hold true?

d. Pick an $x-$value which, according to your drawing, is *outside* the solution set. Plug it into the original inequality $14 - 2x < 20$. Does the inequality hold true?

6. $x - 2y \geq 4$

a. Solve for y.

b. Now—for the moment—let's pretend that your equation said *equals* instead of "greater than" or "less than." Then it would be the equation for a line. Find the slope and the $y-$intercept of that line, and graph it.

Slope: _________

$y-$Intercept: _________

c. Now, pick any point (x, y) that is *above* that line. Plug the x and y coordinates into your inequality from part (a). Does this point fit the inequality? (Show your work...)

d. Now, pick any point (x, y) that is *below* that line. Plug the x and y coordinates into your inequality from part (a). Does this point fit the inequality? (Show your work...)

e. So, is the solution to the inequality the points *below* or *above* the line? Shade the appropriate region on your graph.

Need help? Check out these related videos:

http://www.khanacademy.org/math/algebra/linear-equations-and-inequalitie/v/graphing-linear-inequalities-in-two-variables-2

Some of One, None of the Other
A Solidify Understanding Task

Carlos and Clarita are comparing strategies for writing equations of the boundary lines for the "Pet Sitter" constraints. They are discussing their work on the *space* constraint.

- *Space*: Cat pens will require 6 ft^2 of space, while dog runs require 24 ft^2. Carlos and Clarita have up to 360 ft^2 available in the storage shed for pens and runs, while still leaving enough room to move around the cages.

Carlos' Method: "I made a table. If I don't have any dogs, then I have room for 60 cats. If I use some of the space for 1 dog, then I can have 56 cats. With 2 dogs, I can boards 52 cats. For each additional dog, I can board 4 fewer cats. From my table I know the y-intercept of my line is 60 and the slope is -4, so my equation is
$y = -4x + 60$."

Clarita's Method: "I let x represent the number of dogs, and y the number of cats. Since dog runs require 24 ft^2, $24x$ represents the amount of space used by dogs. Since cat pens require 6 ft^2, $6y$ represents the space used by cats. So my equation is
$24x + 6y = 360$."

1. Since both equations represent the same information, they must be equivalent to each other.

    a. Show the steps you could use to turn Clarita's equation into Carlos' equation. Explain why you can do each step.

    b. Show the steps you could use to turn Carlos' equation into Clarita's. Explain why you can do each step.

2. Use both Carlos' and Clarita's methods to write the equation of the boundary line for the *start-up costs* constraint.

    - *Start-up Costs*: Carlos and Clarita plan to invest much of the $1280 they earned from their last business venture to purchase cat pens and dog runs. It will cost $32 for each cat pen and $80 for each dog run.

3. Show the steps you could use to turn Clarita's *start-up costs* equation into Carlos' equation. Explain why you can do each step.

4. Show the steps you could use to turn Carlos' *start-up costs* equation into Clarita's. Explain why you can do each step.

In addition to writing an equation of the boundary lines, Carlos and Clarita need to graph their lines on a coordinate grid.

Carlos' equations are written in **slope-intercept form**. Clarita's equations are written in **standard form**. Both forms are ways of writing **linear equations**.

Both Carlos and Clarita know they only need to plot two points in order to graph a line.

Carlos' strategy: How might Carlos use his slope-intercept form, $y = -4x + 60$, to plot two points on his line?

Clarita's strategy: How might Clarita use her standard form, $24x + 6y = 360$, to plot two points on her line? (Clarita is really clever, so she looks for the two easiest points she can find.)

Ready, Set, Go!

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Ready

Find a point that satisfies the first equation. Does it also satisfy the second equation?

1. $y = 2x - 3$ and
   $y = -x + 3$

2. $y = 3x + 3$ and
   $y = -x + 3$

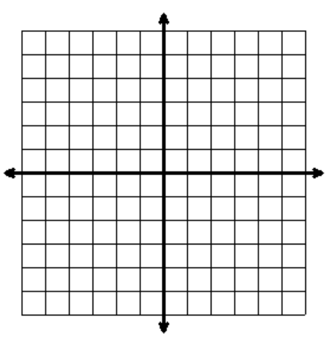
3. $y = 2$ and
   $y = -4x + 3$

4. $y = 2x - 3$ and
   $x + y = -5$

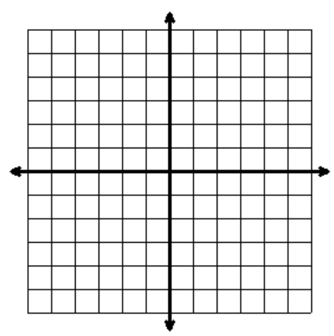
Set

Graph the following equations by finding the intercepts

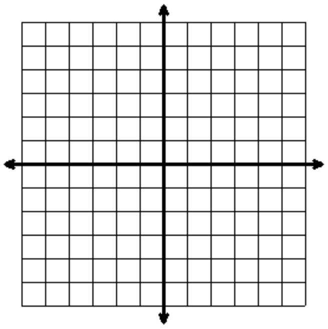
5. $5x - 2y = 15$



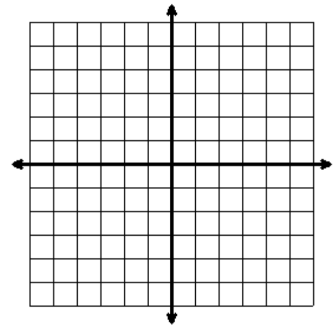
6. $3x + 6y = 25$



7. $6x + y = 3$



8. $x - 8y = 12$



Go

Add. Reduce your answers but leave as improper fractions when applicable.

9. $\frac{3}{4}+\frac{1}{8}$

10. $\frac{3}{5}+\frac{7}{10}$

11. $\frac{2}{3}+\frac{1}{4}$

12. $\frac{4}{7}+\frac{8}{21}$

Multiply. Reduce your answers but leave as improper fractions when applicable.

13. $\frac{3}{4}\bullet\frac{2}{9}$

14. $\frac{4}{7}\bullet\frac{7}{10}$

15. $\frac{2}{9}\bullet\frac{5}{4}$

16. $\frac{3}{7}\bullet\frac{8}{21}$

Need help? Check out these video lessons.

http://www.youtube.com/watch?v=cuNpXve18Pc

http://www.youtube.com/watch?v=6zixwWZ88tk

http://www.youtube.com/watch?v=oHNR0FK_lDE

Pampering and Feeding Time

A Practice Understanding Task

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Carlos and Clarita have been worried about space and start-up costs for their pet sitters business, but they realize they also have a limit on the amount of time they have for taking care of the animals they board. To keep things fair, they have agreed on the following time constraints.

- *Feeding Time*: Carlos and Clarita estimate that cats will require 6 minutes twice a day—morning and evening—to feed and clean their litter boxes, for a total of 12 minutes per day for each cat. Dogs will require 10 minutes twice a day to feed and walk, for a total of 20 minutes per day for each dog. Carlos can spend up to 8 hours each day for the morning and evening feedings, but needs the middle of the day off for baseball practice and games.

- *Pampering Time*: The twins plan to spend 16 minutes each day brushing and petting each cat, and 20 minutes each day bathing or playing with each dog. Clarita needs time off in the morning for swim team and evening for her art class, but she can spend up to 8 hours during the middle of the day to pamper and play with the pets.

Write inequalities for each of these additional time constraints. Shade the solution set for each constraint on separate coordinate grids.

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Ready, Set, Go!

Ready

Topic: Substitution

Determine whether $h = 3$ is a solution to each problem.

1. $3(h-4) = -3$

2. $3h = 2(h+2) - 1$

3. $2h - 3 = h + 6$

4. $3h > -3$

5. $\frac{3}{5} = h \times \frac{1}{5}$

Topic: Solve equations

Determine the value of *x* that makes each equation true.

6. $4x - 2 = 8$

7. $3(x+5) = 20$

8. $2x + 3 = 2x - 5$

Set

Topic: Creating equations, solving real world problems, solve systems of equations

A phone company offers a choice of three text-messaging plans. Plan A gives you unlimited text messages for \$10 a month; Plan B gives you 60 text messages for \$5 a month and then charges you \$0.05 for each additional message; and Plan C has no monthly fee but charges you \$0.10 per message.

9. Write an equation for the monthly cost of each of the three plans.

10. If you send 30 messages per month, which plan is cheapest?

11. What is the cost of each of the three plans if you send 50 messages per month?

12. Determine the values for which each plan is the cheapest?

Go

Topic: Solve literal equations

Re-write each of the following equations for the indicated variable.

13. $3x + 5y = 30$ for y

14. $24x + 6y = 360$ for x

15. $\frac{1280 - 80d}{32} = c$ for d

16. $C = \frac{5}{9}(F - 32)$ for F

17. $y = mx + b$ for b

18. $Ax + By = C$ for y

Need help? Check out these related videos.

What does it mean to be a solution?

http://patrickjmt.com/an-intro-to-solving-linear-equations-what-does-it-mean-to-be-a-solution/

http://patrickjmt.com/solving-linear-equations/

Solving for a variable.

http://www.khanacademy.org/math/algebra/solving-linear-equations/v/solving-for-a-variable

All For One, One For All

A Solidify Understanding Task

Carlos and Clarita have found a way to represent combinations of cats and dogs that satisfy each of their individual "Pet Sitter" constraints, but they realize that they need to find combinations that satisfy all of the constraints simultaneously. Why?

1. Begin by listing the **system of inequalities** you have written to represent the *start-up costs* and *space* "Pet Sitter" constraints.

2. Find at least 5 combinations of cats and dogs that would satisfy both of the constraints represented by this system of inequalities. How do you know these combinations work?

3. Find at least 5 combinations of cats and dogs that would satisfy one of the constraints, but not the other. For each combination, explain how you know it works for one of the inequalities, but not for other?

4. Shade a region on a coordinate grid that would represent the **solution set to the system of inequalities.** Explain how you found the region to shade.

5. Rewrite your systems of inequalities to include the additional constraints for *feeding time* and *pampering time*.

6. Find at least 5 combinations of cats and dogs that would satisfy all of the constraints represented by this new system of inequalities. How do you know these combinations work?

7. Find at least 5 combinations of cats and dogs that would satisfy some of the constraints, but not all of them. For each combination, explain how you know it works for some inequalities, but not for others?

8. Shade a region of a coordinate grid that would represent the solution set to the system of inequalities consisting of all 4 "Pet Sitter" constraints. Explain how you found the region to shade.

9. Shade a region in quadrant 1 of a coordinate grid that would represent all possible combinations of cats and dogs that satisfy the 4 "Pet Sitter" constraints. This set of points is referred to as the **feasible region** since Carlos and Clarita can feasibly board any of the combinations of cats and dogs represented by the points in this region without exceeding any of their constraints on time, money or space.

10. How is the feasible region shaded in #9 different from the solution set to the system of inequalities shaded in #8?

All For One, One For All, part 1

A Solidify Understanding Task

Carlos and Clarita have found a way to represent combinations of cats and dogs that satisfy each of their individual "Pet Sitter" constraints, but they realize that they need to find combinations that satisfy all of the constraints simultaneously. Why?

1. Begin by listing the **system of inequalities** you have written to represent the *start-up costs* and *space* "Pet Sitter" constraints.

2. Find at least 5 combinations of cats and dogs that would satisfy both of the constraints represented by this system of inequalities. How do you know these combinations work?

3. Find at least 5 combinations of cats and dogs that would satisfy one of the constraints, but not the other. For each combination, explain how you know it works for one of the inequalities, but not for other?

4. Shade a region on a coordinate grid that would represent the **solution set to the system of inequalities.** Explain how you found the region to shade.

5. Shade a region in quadrant 1 of a coordinate grid that would represent all possible combinations of cats and dogs that satisfy the start-up costs and space "Pet Sitter" constraints.

6. How is the region shaded in #5 different from the solution set to the system of inequalities shaded in #4?

All For One, One For All, part 2

A Solidify Understanding Task

Carlos and Clarita are trying to find a way to represent combinations of cats and dogs that satisfy all four of their "Pet Sitter" constraints.

So far, they have examined the **system of inequalities** that represents the *start-up costs* and *space* "Pet Sitter" constraints. They shaded a region that represented the solution set to this system of inequalities, and realized that the portion of this shaded region that lies in the first quadrant would contain the points that represent combinations of cats and dogs for which they have space and can afford to purchase pens and runs for their summer business. Now they are wondering how their time constraints will affect the solution set.

1. Rewrite your systems of inequalities to include the additional constraints for *feeding time* and *pampering time*. That is, you should now have a system of inequalities that contains all four constraints.

2. Find at least 5 combinations of cats and dogs that would satisfy all of the constraints represented by this new system of inequalities. How do you know these combinations work?

3. Find at least 5 combinations of cats and dogs that would satisfy some of the constraints, but not all of them. For each combination, explain how you know it works for some inequalities, but not for others?

4. Shade a region on a coordinate grid that would represent the solution set to the system of inequalities consisting of all 4 "Pet Sitter" constraints. Explain how you found the region to shade.

5. Shade a region in quadrant 1 of a coordinate grid that would represent all possible combinations of cats and dogs that satisfy the 4 "Pet Sitter" constraints. This set of points is referred to as the **feasible region** since Carlos and Clarita can feasibly board any of the combinations of cats and dogs represented by the points in this region without exceeding any of their constraints on time, money or space.

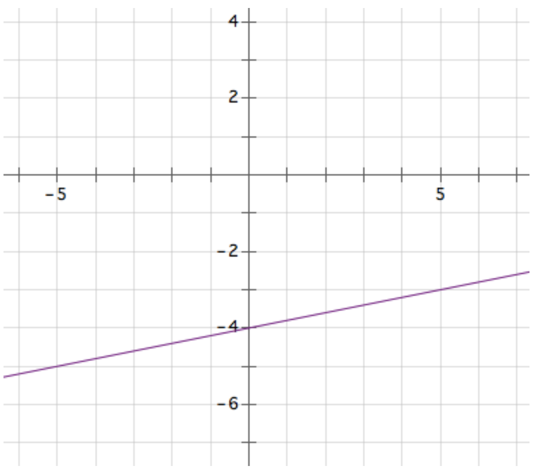
6. How is the feasible region shaded in #5 different from the solution set to the system of inequalities shaded in #4?

Ready, Set, Go!

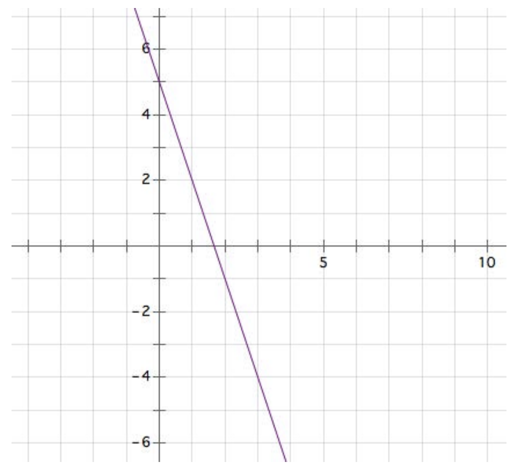
Ready

For each inequality and graph, pick a point and use it to determine which half-plane should be shaded, then shade the correct half-plane.

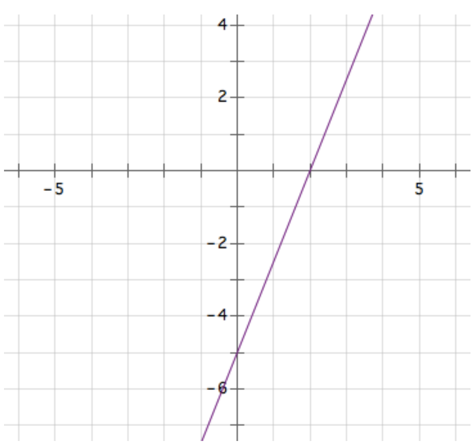
1. $y \le \frac{1}{5}x - 4$



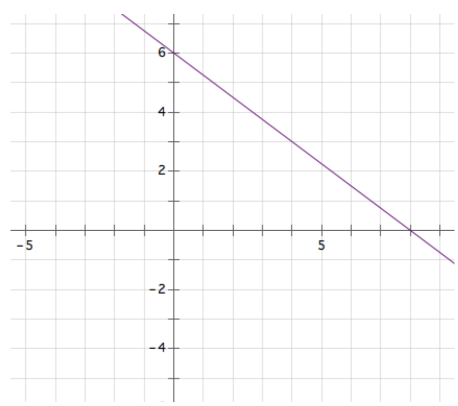
2. $y \ge -3x + 5$



3. $5x - 2y \le 10$



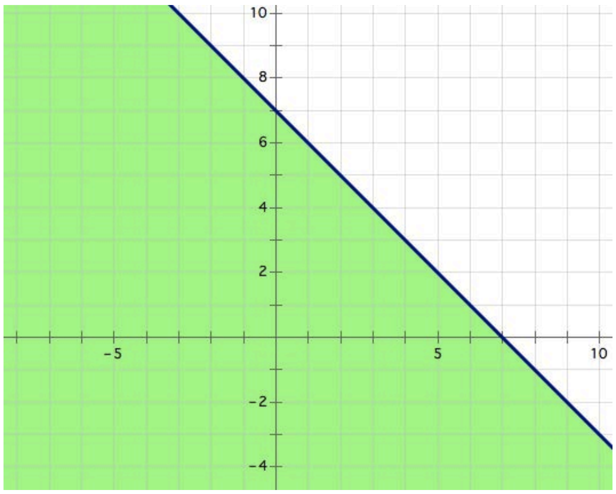
4. $3x + 4y \ge 24$



Set

Given the graph with the regions that are shaded write the inequality or system of inequalities.

5.



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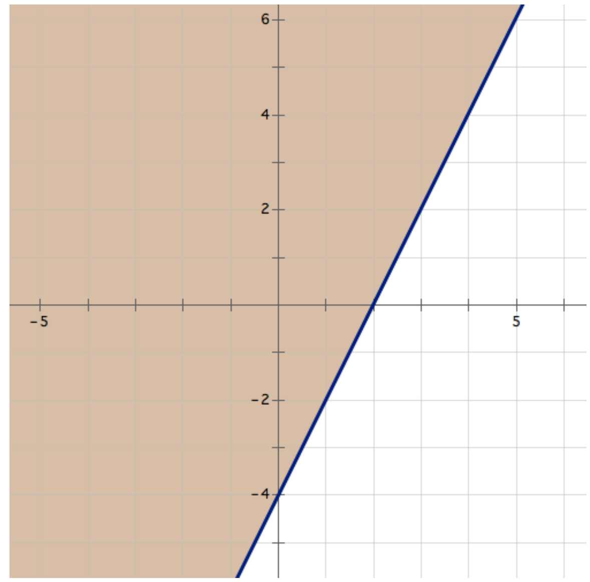

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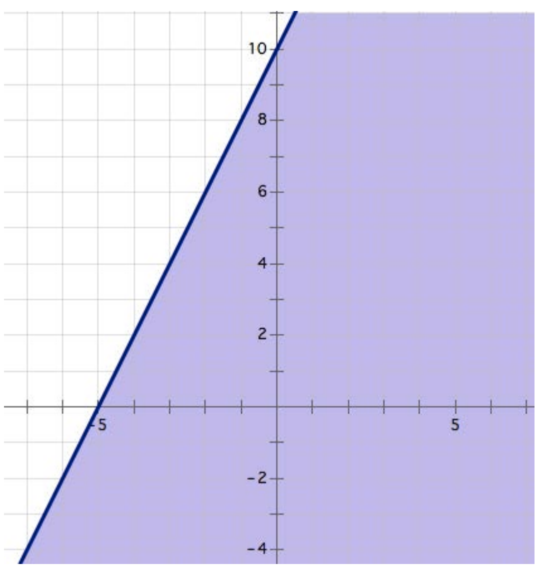
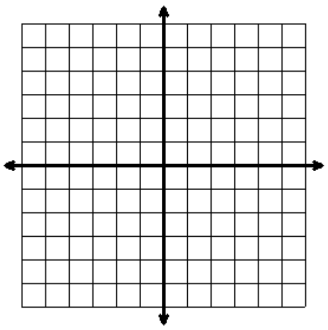


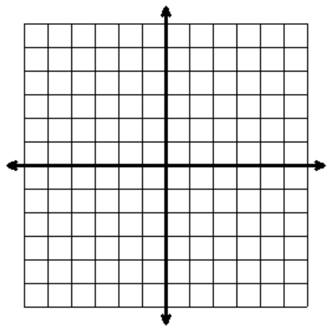
Go

Graph the following inequalities.

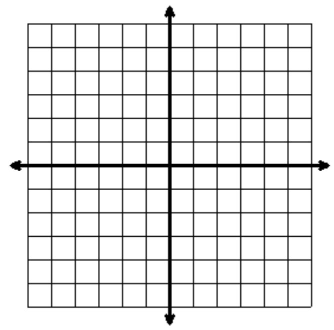
8. $y \le 3x - 4$



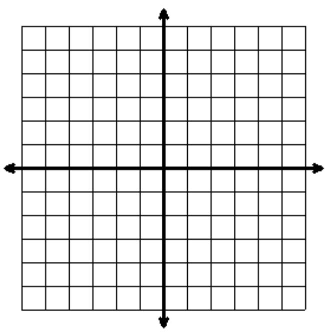
9. $y < -2x + 3$



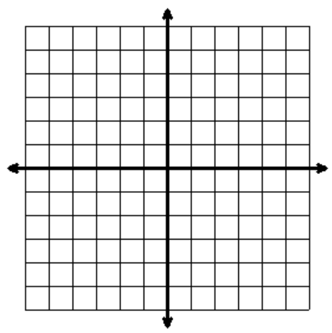
10. $y \ge 4x - 3$



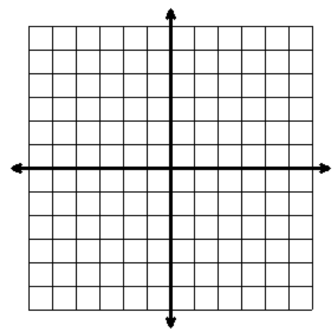
11. $3x + 4y < 12$



12. $5x + 4y \le 25$



13. $6x + 8y \le 24$



Need Help? Check out these related videos.

http://www.khanacademy.org/math/algebra/linear-equations-and-inequalitie/v/graphing-linear-inequalities-in-two-variables-3

Get to the Point!

A Solidify Understanding Task

Carlos and Clarita need to clean the storage shed where they plan to board the pets. They have decided to hire a company to clean the windows. After collecting the following information, they have come to you for help deciding which window cleaning company they should hire.

- *Sunshine Express Window Cleaners* charges \$50 for each service call, plus \$10 per window.
- "*Pane"less Window Cleaners* charges \$25 for each service call, plus \$15 per window.

1. Which company would you recommend, and why? Prepare an argument to convince Carlos and Clarita that your recommendation is reasonable. (It is always more convincing if you can support your claim in multiple ways. How might you support your recommendation using a table? A graph? Algebra?)

Your presentation to Carlos reminds him of something he has been thinking about—how to find the coordinates of the points where the boundary lines in the "Pet Sitter" constraints intersect. He would like to do this algebraically since he thinks guessing the coordinates from a graph might be less accurate.

2. Write equations for the following two constraints.

    - *Space*
    - *Start-up Costs*

    Find where the two lines intersect algebraically. Record enough steps so that someone else can follow your strategy.

3. Now find the point of intersection for the two time constraints.

    - *Feeding Time*
    - *Pampering Time*

Ready, Set, Go!

Ready

Topic: Determine patterns

Find the next two values in the pattern. Describe how you determined these values.

1. 3, 6, 9, 12, _____, _____ description:
2. 3, 6, 12, 24, _____, ___ description:
3. 24, 20, 16, 12, ___, ___ description:
4. 24, 12, 6, 3, ___, ___ description:

Set

Topic: Solve systems of equations using substitution

Solve the system of equations using substitution. Justify graphically.

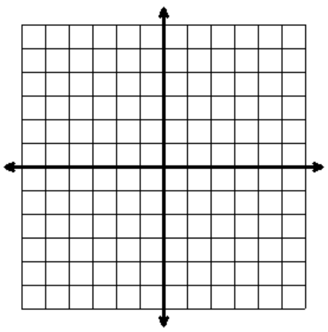
5. $x + 2y = 9$ and
   $3x + 5y = 20$
6. $x - 3y = 10$ and
   $2x + y = 13$
7. $x + 2y = -1$ and
   $3x + 5y = -1$
8. $y = 2x - 3$ and
   $x + y = -5$
9. $x - y = 6$ and
   $3x + y = -5$
10. Tickets to a show cost \$10 in advance and \$15 at the door. If 120 tickets are sold for a total of \$1390, how many of the tickets were bought in advance?

Go

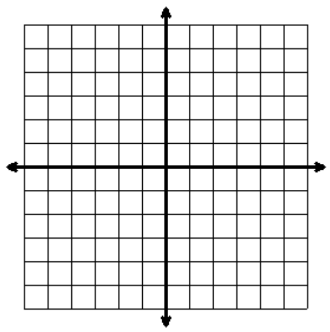
Topic: Graph two variable inequalities

Graph the following inequalities.

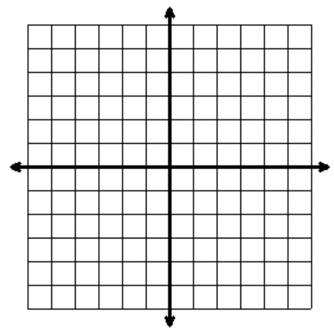
8. $y \leq 3x - 4$



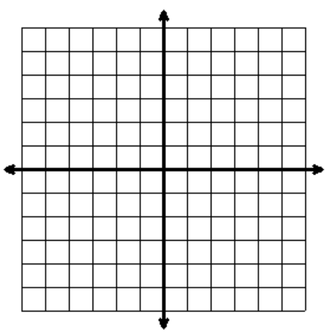
9. $y \leq -2x + 3$



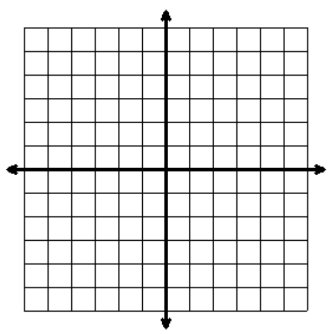
10. $y \geq 4x - 3$



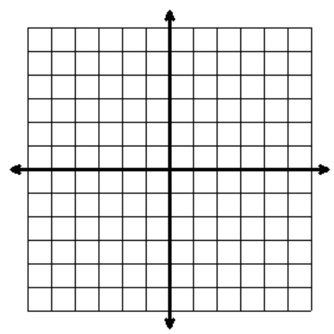
11. $3x + 4y < 12$



12. $5x + 4y \leq 25$



13. $6x + 8y \leq 24$



Need help? Check out these related videos.

http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/solving-systems-by-substitution-3

http://www.khanacademy.org/math/algebra/linear-equations-and-inequalitie/v/solving-and-graphing-linear-inequalities-in-two-variables-1

http://www.khanacademy.org/math/algebra/linear-equations-and-inequalitie/v/graphing-inequalities-2

Shopping for Cats and Dogs
A Develop Understanding Task

Clarita is upset with Carlos because he has been buying cat and dog food without recording the price of each type of food in their accounting records. Instead, Carlos has just recorded the total price of each purchase, even though the total cost includes more that one type of food. Carlos is now trying to figure out the price of each type of food by reviewing some recent grocery receipts. See if you can help him figure out the cost of particular items on the receipts, and be prepared to explain your reasoning to Carlos. (For each of the following scenarios, assume that these are the purchase prices without sales tax.)

1. One week Carlos bought 3 bags of *Tabitha Tidbits* and 4 bags of *Figaro Flakes* for $43.00. The next week he bought 3 bags of *Tabitha Tidbits* and 6 bags of *Figaro Flakes* for $54.00. Based on this information, can you figure out the price of one bag of each type of cat food? Explain your reasoning.

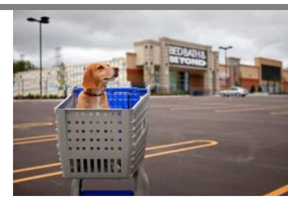
2. One week Carlos bought 2 bags of *Brutus Bites* and 3 bags of *Lucky Licks* for $42.50. The next week he bought 5 bags of *Brutus Bites* and 6 bags of *Lucky Licks* for $94.25. Based on this information, can you figure out the price of one bag of each type of dog food? Explain your reasoning.

3. Carlos purchased 6 dog leashes and 6 cat brushes for $45.00 for Clarita to use while pampering the pets. Later in the summer he purchased 3 additional dog leashes and 2 cat brushes for $19.00. Based on this information, can you figure out the price of each item? Explain your reasoning.

4. One week Carlos bought 2 packages of dog bones and 4 packages of cat treats for $18.50. Because the finicky cats didn't like the cat treats, the next week Carlos returned 3 unopened packages of cat treats and bought 2 more packages of dog bones. After being refunded for the cat treats, Carlos only had to pay $1.00 for his purchase. Based on this information, can you figure out the price of each item? Explain your reasoning.

5. Carlos has noticed that because each of his purchases have been somewhat similar, it has been easy to figure out the cost of each item. However, his last set of receipts has him puzzled. One week he tried out cheaper brands of cat and dog food. On Monday he purchased 3 small bags of cat food and 5 small bags of dog food for $22.75. Because he went through the small bags quite quickly, he had to return to the store on Thursday to buy 2 more small bags of cat food and 3 more small bags of dog food, which cost him $14.25. Based on this information, can you figure out the price of each bag of the cheaper cat and dog food? Explain your reasoning.

Summarize the strategies you have used to reason about the price of individual items in the problems given above. What are some key ideas that seem helpful?



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Ready, Set, Go!

Ready

Topic: Exponents

Write in exponential notation:

1. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

2. $3x \cdot 3x \cdot 3x$

Find each value.

3. 2^3

4. 3^3

5. 2^5

6. $(-2)^3$

7. 4^3

Set

Topic: Solving systems

8. Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for \$2.84. Peter also buys three candy bars, but can only afford one additional fruit roll-up. His purchase costs \$1.79. What is the cost of a candy bar and a fruit roll-up individually?

9. A farmer noticed that his chickens were loose and were running around with the cows in the cow pen. He quickly counted 100 heads and 270 legs. How many chickens did he have and how many cows?

Go

Topic: Solve one variable inequalities.

Solve the following inequalities. Write the solution set in interval notation and graph the solution set on a number line.

10. $4x + 10 < 2x + 14$

11. $2x + 6 > 55 - 5x$

12. $2(\frac{x}{4} + 3) > 6(x - 1)$

13. $9x + 4 \leq -2\left(x + \frac{1}{2}\right)$

Solve each inequality. Give the solution in inequality notation and set notation.

14. $\frac{x}{-3} > -\frac{10}{9}$

15. $5x > 8x + 27$

16. $\frac{x}{4} > \frac{5}{4}$

17. $3x - 7 \geq 3(x - 7)$

18. $2x < 7x - 36$

19. $5 - x < 9 + x$

Need help? Check out these related videos?

Exponential notation: http://www.khanacademy.org/math/algebra/exponents-radicals/v/understanding-exponents

Solving inequalities: http://www.khanacademy.org/math/algebra/solving-linear-inequalities/v/solving-inequalities

http://www.khanacademy.org/math/algebra/solving-linear-inequalities/v/multi-step-inequalities-2

Set notation and interval notation: http://patrickjmt.com/using-interval-notation-to-express-inequalities-ex1/

Can You Get to the Point, Too?

A Solidify Understanding Task

Part 1

In "Shopping for Cats and Dogs," Carlos found a way to find the cost of individual items when given the purchase price of two different combinations of those items. He would like to make his strategy more efficient by writing it out using symbols and algebra. Help him formalize his strategy by doing the following:

- For each scenario in "Shopping for Cats and Dogs" write a **system of equations** to represent the two purchases.
- Show how your strategies for finding the cost of individual items could be represented by manipulating the equations in the system. Write out intermediate steps symbolically, so that someone else could follow your work.
- Once you find the price of one of the items in the combination, show how you would find the price of the other item.

Part 2

Writing out each system of equations reminded Carlos of his work with solving systems of equations graphically. Show how each scenario in "Shopping for Cats and Dogs" can be represented graphically, and how the cost of each item shows up in the graphs.

Part 3

Carlos also realized that the algebraic strategy he created in part 1 could be used to find the points of intersection for the "Pet Sitters" constraints. Use the **elimination of variables** method developed in part 1 to find the point of intersection for each of the following pairs of "Pet Sitter" constraints.

- *Start-up costs* and *space* constraints
- *Pampering time* and *feeding time* constraints
- Any other pair of "Pet Sitter" constraints of your choice

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Ready, Set, Go!

Ready

Topic: Evaluate exponents

Simplify and evaluate the following.

1. 3^{-2}
2. $(0.5)^{-2}$
3. 2^4
4. 4^{-2}

Write the following expression three different ways (one way can include the simplified value).

5. $(2^3)(4)$

6. $(3^3)(2^3)$

Set

Topic: Solve systems of equations

Solve the following systems of equations using elimination of variables, then justify graphically.

7. Solve the system: $\begin{aligned} 2x + 0.5y &= 3 \\ x + 2y &= 8.5 \end{aligned}$

8. Solve the system: $\begin{aligned} 3x + 5y &= -1 \\ x + 2y &= -1 \end{aligned}$

9. Solve the system: $\begin{aligned} 3x + 5y &= -3 \\ x + 2y &= -\tfrac{4}{3} \end{aligned}$

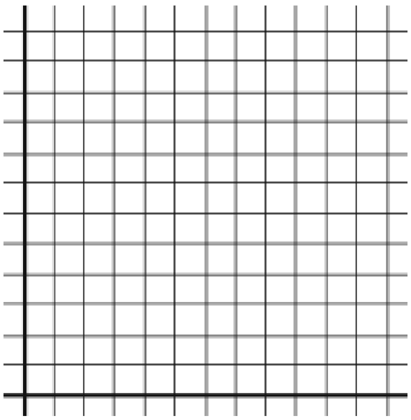
10. A 150-yard pipe is cut to provide drainage for two fields. If the length of one piece is three yards less that twice the length of the second piece, what are the lengths of the two pieces?

Go

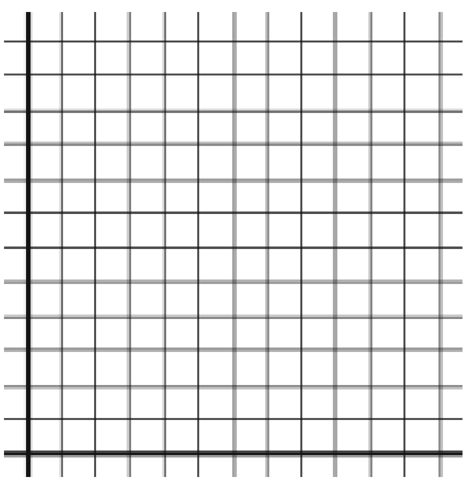
Topic: Graph two variable linear inequalities

Graph the following linear inequalities on the graphs below. Include constraints.

11. Ben has enough money to buy up to eight yogurts. If his favorite flavors are blueberry and strawberry, what are all the possible combinations he can buy? Graph the inequality that shows all possible combinations of his favorite flavors.



12. Peggy is buying a balloon bouquet. Her favorite colors are silver and purple. The silver balloons are $1 and the purple balloons are $0.80. Graph an inequality that shows how many of each color balloon she can put in her bouquet if she doesn't spend more than $20.



Need help? Check out these related videos.

Negative exponents

http://patrickjmt.com/negative-exponents/

http://www.khanacademy.org/math/algebra/ck12-algebra-1/v/zero--negative--and-fractional-exponents

Solving systems by elimination

http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/solving-systems-by-elimination-2

Solving systems by graphing

http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/solving-linear-systems-by-graphing

Food for Fido and Fluffy
A Solidify Understanding Task

Carlos and Clarita have found two different cat foods that seem to appeal to even the most finicky of cats: *Tabitha Tidbits* and *Figaro Flakes.* Each ounce of *Tabitha Tidbits* contains 2 grams of protein, 4 grams of carbohydrates and 4 grams of fat. Each ounce of *Figaro Flakes* contains 3 grams of protein, 4 grams of carbohydrates and 2 grams of fat. Since *Tabitha Tidbits* is fairly expensive, while *Figaro Flakes* is very cheap, the twins have decided to create a new cat food by mixing the two. After studying some nutritional guidelines for cats, Carlos and Clarita have decided to create a mixture based on the following constraints.

- *Amount of Protein*: Each meal should contain at least 12 grams of protein.
- *Amount of Carbohydrates:* Each meal should contain more than 16 grams of carbohydrates.
- *Amount of Fats:* Each meal should contain no more than 18 grams of fat.
- *Size of a Feeding:* Each meal should consist of less than 10 ounces of food.

For the work that follows, let *T* represent the number of ounces of *Tabitha Tidbits* in a meal and let *F* represent the number of ounces of *Figaro Flakes.*

1. Write an inequality for each of the constraints.

2. On separate coordinate grids, graph the solution set for each of the inequalities you wrote in #1. How do you know on which side of the boundary line you should shade the half-plane that represents the solution set?

3. Decide if the boundary line for each inequality represented in #2 should be a solid line or a dotted line. Which words or phrases in the constraints suggested a solid line? A dotted line?

4. Find at least 5 combinations of *Tabitha Tidbits* and *Figaro Flakes* Carlos and Clarita can mix together to create a nutritious cat meal. Show that these points lie within a feasible region for these constraints.

5. *Brutus Bites* is a brand of dog food that contains 4 grams of protein and 6 grams of fat per ounce. *Lucky Licks* is another brand of dog food that contains 12 grams of protein and 4 grams of fat per ounce. Carlos wants to make a meal for dogs that contains at least 8 grams of protein and no more than 6 grams of fat. Write and solve a system of inequalities that Carlos can use to determine a combination of *Brutus Bites* and *Lucky Licks* that will satisfy these constraints.

Ready, Set, Go!

Ready

Topic: Solving two variable inequalities

1. A theater wants to take in at least $2000 for a certain matinee. Children's tickets cost $5 each and adult tickets cost $10 each.

    a. Write an inequality describing the number of tickets that will allow the theater to meet their goal of $2000.

    b. If the theater has a maximum of 350 seats, write an inequality describing the number of both types of tickets the theater can sell.

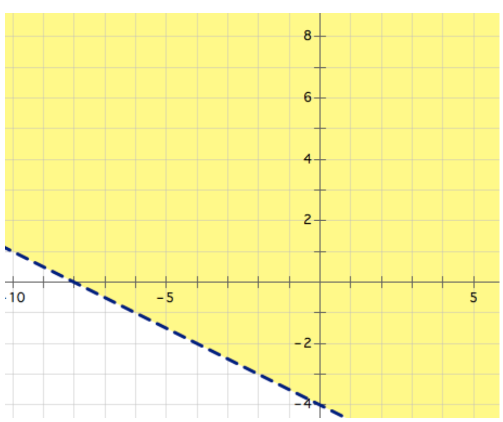
    c. Find the number of children and adult tickets that can be sold so that all seats are sold and the $2000 goal is reached.

Set

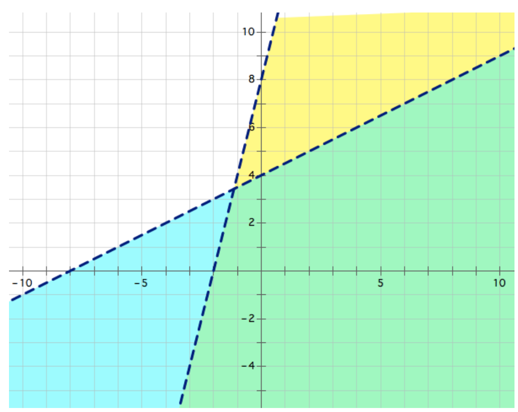
Topic: Writing equations of two variable inequalities

Given the graph with the regions that are shaded write the inequality or system of inequalities.

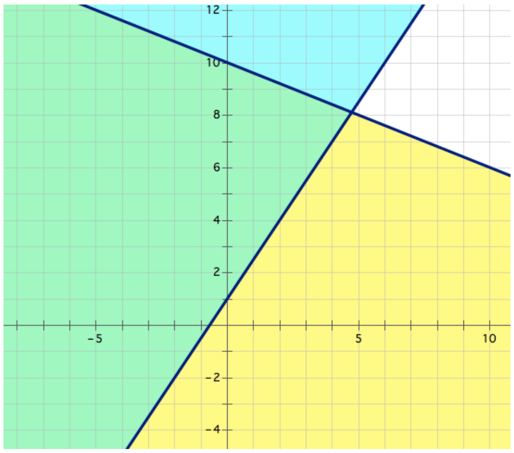
2.



3.



4.



Go

Topic: Graph two variable inequalities

Graph each set of inequalities below. Include the shaded region of both, plus indicate the region that is true for all inequalities.

5. $$\begin{aligned} x - y &< -6 \\ 2y &\geq 3x + 17 \end{aligned}$$

6. $$\begin{aligned} 5x - y &\geq 5 \\ 2y - x &\geq -10 \end{aligned}$$

Solve the following systems of equations.

7. Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for $2.84. Peter also buys three candy bars, but can only afford one additional fruit roll-up. His purchase costs $1.79. What is the cost of a candy bar and a fruit roll-up individually?

10. $$\begin{aligned}5x - 10y &= 15\\ 3x - 2y &= 3\end{aligned}$$

11. $$\begin{aligned}5x - y &= 10\\ 3x - 2y &= -1\end{aligned}$$

12. $$\begin{aligned}5x + 7y &= 15\\ 7x - 3y &= 5\end{aligned}$$

Need help? Check out these related videos.

http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/graphing-systems-of-inequalities-2

Taken Out of Context

A Practice Understanding Task

Write a shopping scenario similar to those in "Shopping for Cats and Dogs" to fit each of the following systems of equations. Then use the elimination of variables method you invented in "Can You Get to the Point, Too" to solve the system. Some of the systems may have interesting or unusual solutions. See if you can explain them in terms of the shopping scenarios you wrote.

1. $\begin{cases} 3x + 4y = 23 \\ 5x + 3y = 31 \end{cases}$

2. $\begin{cases} 2x + 3y = 14 \\ 4x + 6y = 28 \end{cases}$

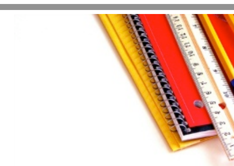
3. $\begin{cases} 3x + 2y = 20 \\ 9x + 6y = 35 \end{cases}$

4. $\begin{cases} 4x + 2y = 8 \\ 5x + 3y = 9 \end{cases}$

Three of Carlos' and Clarita's friends are purchasing school supplies at the bookstore. Stan buys a notebook, three packages of pencils and two markers for $7.50. Jan buys two notebooks, six packages of pencils and five markers for $15.50. Fran buys a notebook, two packages of pencils and two markers for $6.25. How much do each of these three items cost?

Explain in words or with symbols how you can use your intuitive reasoning about these purchases to find the price of each item.

Ready, Set, Go!



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Ready

For each of the systems of inequalities, determine if the given coordinates are solutions to the system.

1. $\begin{cases} y \le 3x - 5 \\ y \ge x + 2 \end{cases}$

   a. (6 , 10)

   b. (1 , 4)

   c. (8 , 15)

2. $\begin{cases} y > -2x + 9 \\ y \ge 5x - 6 \end{cases}$

   a. (-2 , -5)

   b. (-1 , 12)

   c. (5 , 0)

3. $\begin{cases} y < -\dfrac{1}{2}x + 9 \\ y > 6x - 10 \end{cases}$

   a. (-2 , -5)

   b. (7 , 3)

   c. (-8 , 10)

Set

Topic: Determine the number of solutions in a system of equations

Express each equation in slope-intercept form. Without graphing, state whether the system of equations has zero, one or infinite solutions (consistent, inconsistent or dependent.). How do you know?

4. $\begin{array}{l} 3x - 4y = 13 \\ y = -3x - 7 \end{array}$

5. $\begin{array}{l} 3x - 3y = 3 \\ x - y = 1 \end{array}$

6. $\begin{array}{l} 0.5x - y = 30 \\ 0.5x - y = -30 \end{array}$

7. $\begin{array}{l} 4x - 2y = -2 \\ 3x + 2y = -12 \end{array}$

Go

Topic: Graph two variable inequalities

Graph the following inequalities. Be sure to label your axes and scale.

Justify the region you shade by showing three points in the region as being solutions to the problem. Show a point you have tested to prove your shaded region is accurate.

8. $3x - 4y \geq 12$

9. $x + 7y < 5$

10. $6x + 5y > 1$

11. $x - \frac{1}{2}y \geq 5$

12. $6x + y < 20$

13. $30x + 5y < 100$

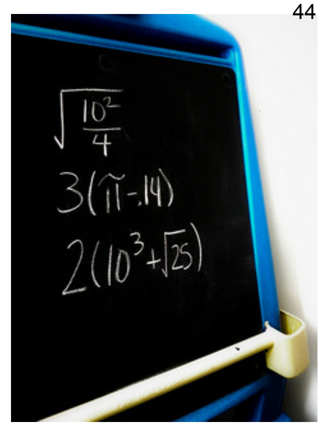
14. On the same set of axes, graph $y > x + 2$ and $y < x + 5$. What values do these two have in common?

Need help? Check out these related videos

Testing a solution to an equation http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/testing-a-solution-for-a-system-of-equations

Number of solutions http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/special-types-of-linear-systems

Solving inequalities http://www.khanacademy.org/math/algebra/solving-linear-inequalities/v/solving-inequalities



More Things Taken Out of Context

A Practice Understanding Task

Solve the following systems of inequalities:

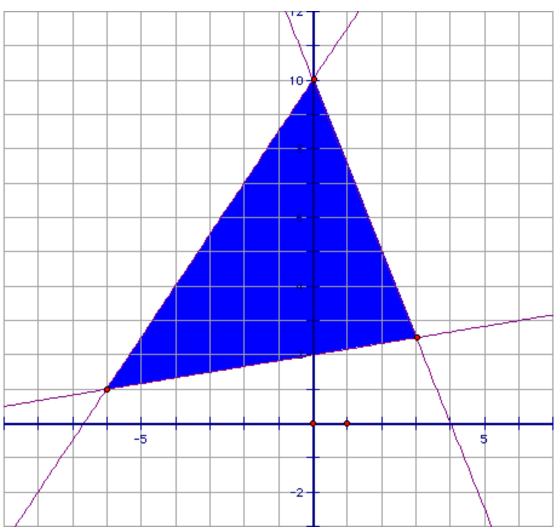
1. $\begin{cases} -5x+3y \le 45 \\ 2x+3y > 24 \end{cases}$

2. $\begin{cases} -10x+6y \le 90 \\ 6x+9y > 36 \end{cases}$

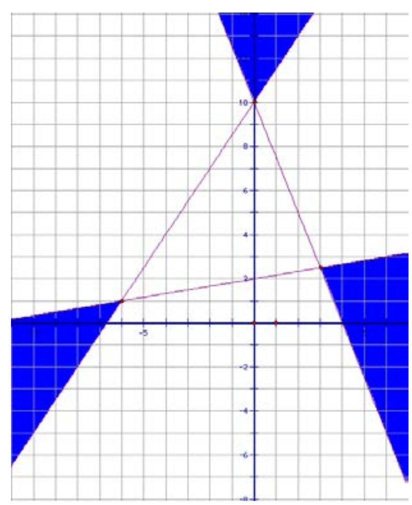
3. Is the point (-3, 10) a solution to the system in problem #1? Why or why not?

4. How are the inequalities representing the boundaries of the solution sets in problems #1 and #2 similar to each other? What accounts for these similarities?

5. Write the system of inequalities whose solution set is shown below:



6. Amanda is examining Frank's work on #5, when she exclaims, "You have written all of your inequalities backwards. The solution set to your system would look like this."



What do you think about Amanda's statement?

Ready, Set, Go!

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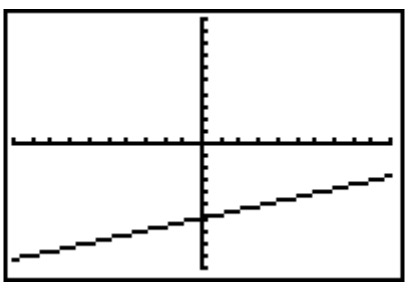
Ready

Topic: Determine a good viewing window for graphs

When sketching a graph of a function, it is important that we see important points. For linear functions, we want a window that shows important information related to the story. Often, this means including both the x- and y- intercepts.

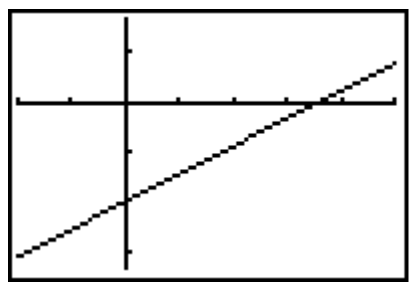
Example: $g(x) = \frac{1}{3}x - 6$

Window: [-10, 10] by [-10,10]
x- scale: 1 y-scale: 1



NOT a good window

Window: [-10, 25] by [-10, 5]
x-scale: 5 y-scale: 5



Good window

For the following equations, state a window that would be satisfactory for the given equation. Then sketch a graph in the boxes provided. If using a scale other than one, make sure to indicate this on your graph.

1. $f(x) = 3x - 100$

[ ] by [ ]
x-scale: y-scale:

2. $5x + 7y = 15$

[ ] by [ ]
x-scale: y-scale:

3. $y = 5x + 15$

[ ] by [ ]
x-scale: y-scale:

4. $y = \frac{1}{3}x - 20$

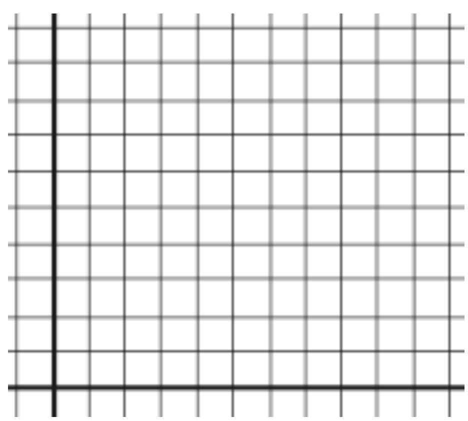
[ ] by [ ]
x-scale: y-scale:

Set

Topic: Creating and solving two variable inequalities

5. Patty makes $8 per hour mowing lawns and $12 per hour babysitting. She wants to make at least $100 per week but can work no more than 12 hours a week. Write and graph a system of linear inequalities.

List 2 possible combinations of hours that Patty could work at each job.



Go

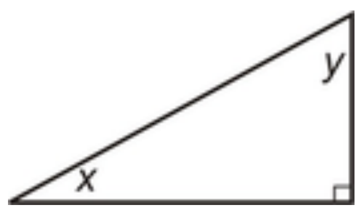
Topic: Solve systems of equations

Solve each system of equations

6. $$\begin{aligned} 3x + 5y &= -3 \\ x + 2y &= -\tfrac{4}{3} \end{aligned}$$

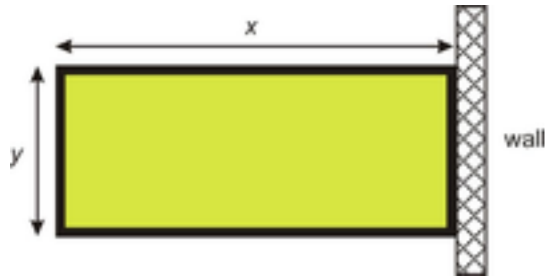
7. $$\begin{aligned} x - y &= -\tfrac{12}{5} \\ 2x + 5y &= -2 \end{aligned}$$

8. Of the two non-right angles in a right triangle, one measures twice as many degrees as the other. What are the angles?

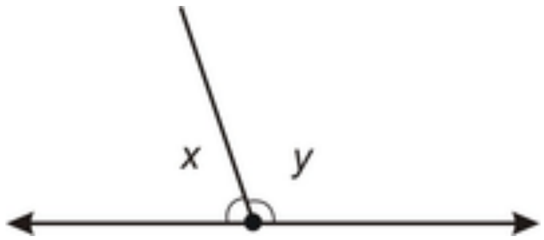


9. The sum of two numbers is 70 and the difference is 11. What are the numbers?

10. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?



11. A ray cuts a line forming two angles. The difference between the two angles is 18°. What does each angle measure?



Need Help? Check out these related videos:

http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/system-of-inequalities-application

Pet Sitters Revisited

A Develop Understanding Task

Carlos and Clarita have successfully found a way to represent *all* of the combinations of cats and dogs that they can board based on *all* of the following constraints.

- *Space*: Cat pens will require 6 ft^2 of space, while dog runs require 24 ft^2. Carlos and Clarita have up to 360 ft^2 available in the storage shed for pens and runs, while still leaving enough room to move around the cages.

- *Feeding Time*: Carlos and Clarita estimate that cats will require 6 minutes twice a day—morning and evening—to feed and clean their litter boxes, for a total of 12 minutes per day for each cat. Dogs will require 10 minutes twice a day to feed and walk, for a total of 20 minutes per day for each dog. Carlos can spend up to 8 hours each day for the morning and evening feedings, but needs the middle of the day off for baseball practice and games.

- *Pampering Time*: The twins plan to spend 16 minutes each day brushing and petting each cat, and 20 minutes each day bathing or playing with each dog. Clarita needs time off in the morning for swim team and evening for her art class, but she can spend up to 8 hours during the middle of the day to pamper and play with the pets.

- *Start-up Costs*: Carlos and Clarita plan to invest much of the $1280 they earned from their last business venture to purchase cat pens and dog runs. It will cost $32 for each cat pen and $80 for each dog run.

Now they are trying to determine how many of each type of pet they should plan to accommodate. Of course, Carlos and Clarita want to make as much money as possible from their business, so they need to pay attention to both their daily income as well as their daily costs. They plan to charge $8 per day for boarding each cat and $20 per day for each dog. They estimate that each cat will require $2.00 per day in food and supplies, and that each dog will require $4.00 per day in costs.

After surveying the community regarding the pet boarding needs, Carlos and Clarita are confident that they can keep all of their boarding spaces filled for the summer.

So the question is, how many of each type of pet should they prepare for in order to make as much money as possible?

What combination of cats and dogs do you think will make the most money? What recommendations would you give to Carlos and Clarita, and what argument would you use to convince them that your recommendation is reasonable?

To get started on this task, you might want to look for collections of points where the daily profit is the same. For example, can you find a collection of points where for each point the daily profit is $120? What about $180?

Ready, Set, Go!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Ready

Topic: Solve exponential equations

Find the value of *x* for each situation.

1. $2^x = 8$ 2. $3^x = 27$ 3. $2^x = 4$ 4. $(-2)^x = -8$

Set

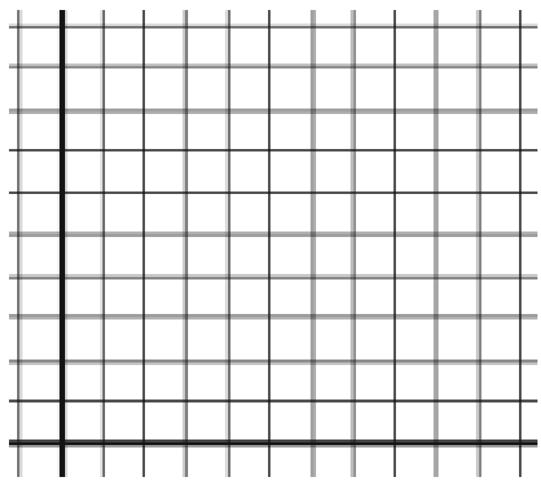
Topic: Create and solve two variable inequalities

5. Jane is buying fruit salad and potato salad for a picnic. Fruit salad costs \$2.00 per pound and potato salad costs \$4.00 per pound. Jane needs to buy at least 6 pounds of salads and she doesn't want to spend more than \$20. Write and graph a system of linear inequalities.

Let x = pounds of fruit salad.

Let y = pounds of potato salad.

List 2 possible combinations of salad that Jane could buy.



Go

Topic: Find the solution region of the following systems of inequalities.

Graph each set of inequalities and determine the solution region

6. $$\begin{aligned} x - y &< -6 \\ 2y &\geq 3x + 17 \end{aligned}$$

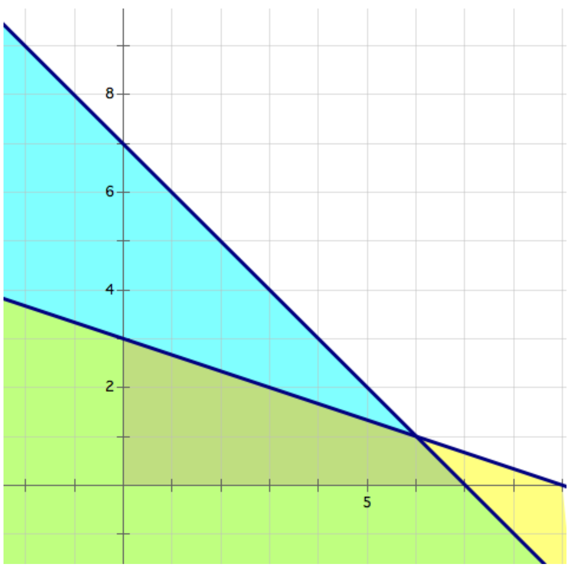
7. $$\begin{aligned} 4y - 5x &< 8 \\ -5x &\geq 16 - 8y \end{aligned}$$

8. $$\begin{aligned} 5x - y &\geq 5 \\ 2y - x &\geq -10 \end{aligned}$$

9. $$\begin{aligned} 5x + 2y &\geq -25 \\ 3x - 2y &\leq 17 \\ x - 6y &\geq 27 \end{aligned}$$

10. $$\begin{aligned} 2x - 3y &\leq 21 \\ x + 4y &\leq 6 \\ 3x + y &\geq -4 \end{aligned}$$

11. Write the inequalities that would create the graph below.



Need help? Check out these related videos.

Exponents http://patrickjmt.com/exponents-intro-to-evaluating-a-few-truefalse-questions/

Rules for exponents http://patrickjmt.com/basic-exponent-properties/

Solving a system of inequalities http://www.khanacademy.org/math/algebra/ck12-algebra-1/v/systems-of-linear-inequalities

To Market with Matrices
A Solidify Understanding Task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Carlos learned about matrices when Elvira, the manager of the school cafeteria, was asked to substitute teach during one of the last days of school before summer vacation. Now that he has worked out a strategy for solving systems of equations by elimination of variables, he is wondering if matrices can help him keep track of his work.

Carlos is reconsidering the following scenario from "Shopping for Cats and Dogs", while trying to record his thinking using matrices.

> *One week Carlos purchased 6 dog leashes and 6 cat brushes for $45.00 for Clarita to use while pampering the pets. Later in the summer he purchased 3 additional dog leashes and 2 cat brushes for $19.00. What is the price of each item?*

Carlos realizes that he can represent this scenario using the following matrix:

$$\begin{array}{r c} & \begin{array}{ccc} \textit{leashes} & \textit{brushes} & \textit{total} \end{array} \\ \begin{array}{r} \textit{purchase } 1 \\ \textit{purchase } 2 \end{array} & \begin{bmatrix} 6 & 6 & \$45.00 \\ 3 & 2 & \$19.00 \end{bmatrix} \end{array}$$

He also realizes that he can represent the cost of each item with a matrix that looks like this:

$$\begin{array}{r c} & \begin{array}{ccc} \textit{leashes} & \textit{brushes} & \textit{total} \end{array} \\ \begin{array}{r} \textit{purchase } 1 \\ \textit{purchase } 2 \end{array} & \begin{bmatrix} 1 & 0 & \$4.00 \\ 0 & 1 & \$3.50 \end{bmatrix} \end{array}$$

So, now he is trying to find a sequence of matrices that can fill in the gaps between the first matrix and the last. He knows from his previous work with solving systems of equations that he can do any of the following manipulations with equations—and he realizes that each of these manipulations would give him a new row of numbers in a corresponding matrix.

- Replace an equation in the system with a constant multiple of that equation
- Replace an equation in the system with the sum or difference of the two equations
- Replace an equation with the sum of that equation and a multiple of the other

1. Help Carlos find a sequence of matrices that starts with the matrix that represents the original purchases, and ends with the matrix that represents purchasing one leash or purchasing one brush. For each matrix in your sequence, write out the justification that allows you to write that matrix based on the three manipulations we can perform on the equations in a system. For example, the following matrix transformation can be justified by writing "I replaced the first row of the matrix by multiplying the first row by $\frac{1}{6}$."

$$\begin{bmatrix} 6 & 6 & 45.00 \\ 3 & 2 & 19.00 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 7.50 \\ 3 & 2 & 19.00 \end{bmatrix}$$

2. Find and justify a sequence of matrices that could be used to solve the following scenario.

One week Carlos tried out cheaper brands of cat and dog food. On Monday he purchased 3 small bags of cat food and 5 small bags of dog food for $22.75. Because he went through the small bags quite quickly, he had to return to the store on Thursday to buy 2 more small bags of cat food and 3 more small bags of dog food, which cost him $14.25. Based on this information, can you figure out the price of each bag of the cheaper cat and dog food?

Ready, Set, Go!

Ready

Topic: Solving Systems by Substitution and Elimination

Solve each system of equations using an algebraic method.

1. $\begin{cases} 3x - y = 1 \\ 3x + 2y = 16 \end{cases}$

2. $\begin{cases} x + 2y = 5 \\ 3x + 5y = 14 \end{cases}$

3. $\begin{cases} 4x + 2y = -8 \\ x - 2y = -7 \end{cases}$

4. $\begin{cases} 2x + 3y = 2 \\ 3x - 4y = -14 \end{cases}$

5. $\begin{cases} x + 2y = 11 \\ x - 4y = 2 \end{cases}$

6. $\begin{cases} 2x + y = 0 \\ 5x + 3y = 1 \end{cases}$

Set

Topic: Row reductions in Matrices

Create a matrix to match each step in the solving of the system of equations given.

Also, write what happened to the equation and the matrix between steps.

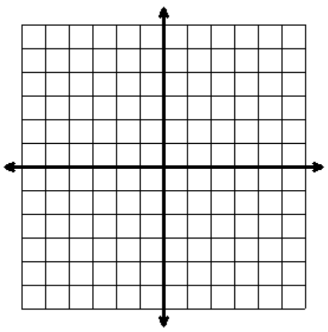
7.

	System	Matrix	Verbal Description
Given System	$\begin{cases} 3x + 2y = 40 \\ x - 7y = -2 \end{cases}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	
Step 1	$\begin{cases} 3x + 2y = 40 \\ -3x + 21y = 6 \end{cases}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	
Step 2	$\begin{cases} 3x + 2y = 40 \\ 0x + 23y = 46 \end{cases}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	
Step 3	$\begin{cases} 3x + 2y = 40 \\ 0x + y = 2 \end{cases}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	
Step 4	$\begin{cases} 3x + 0y = 36 \\ 0x + y = 2 \end{cases}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	
Step 5	$\begin{cases} x + 0y = 12 \\ 0x + y = 2 \end{cases}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	

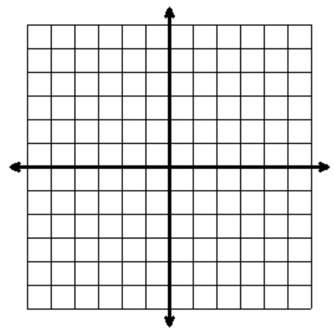
Go

Solve each system of equations by graphing.

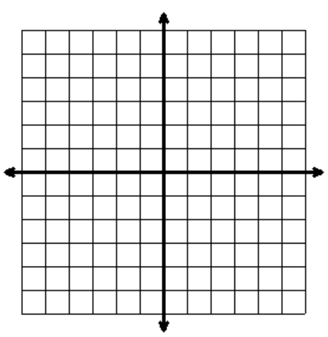
8. $\begin{cases} y = 3x - 3 \\ y = -3x + 9 \end{cases}$



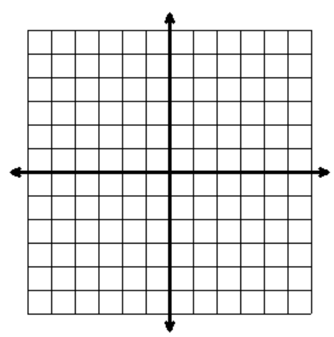
9. $\begin{cases} y = 4x - 1 \\ y = -x + 4 \end{cases}$



10. $\begin{cases} y = -2x + 7 \\ -3x + y = -8 \end{cases}$



11. $\begin{cases} 4x - y = 7 \\ -6x + 2y = 8 \end{cases}$



Need help? Check out these related videos:

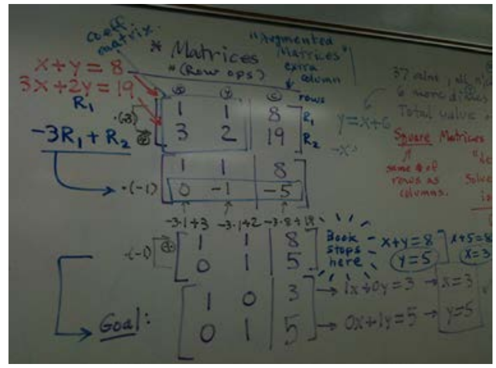
http://www.khanacademy.org/math/algebra/ck12-algebra-1/v/solving-linear-systems-by-substitution

http://patrickjmt.com/row-reducing-a-linear-system-of-equations/

http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/graphings-systems-of-equations

Solving Systems with Matrices

A Practice Understanding Task



In the task "To Market with Matrices" you developed a strategy for solving systems of linear equations using matrices. An efficient and consistent way to carry out this strategy can be summarized as follows:

To row reduce a matrix:

- Perform elementary row operations to yield a "1" in the first row, first column.
- Create zeros in all of the other rows of the first column by adding the first row times a constant to each other row.
- Perform elementary row operations to yield a "1" in the second row, second column.
- Create zeros in all of the other rows of the second column by adding the second row times a constant to each other row.
- Perform elementary row operations to yield a "1" in the third row, third column.
- Create zeros in all of the other rows of the third column by adding the third row times a constant to each other row.
- Continue this process until the first m × m entries form a square matrix with 1s in the diagonal and 0s everywhere else.

Practice this strategy by creating a sequence of matrices for each of the following that begins with the given matrix and ends with the left portion of the matrix (the first m × m entries) in row-reduced form. Write a description of what you did to get from one matrix to another in each step of your sequence of matrices.

1. $\begin{bmatrix} 2 & 4 & 0 \\ 3 & 5 & -2 \end{bmatrix}$

2. $\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 11 \end{bmatrix}$

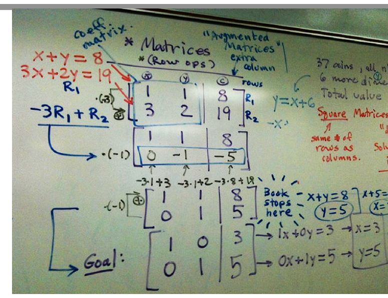
3. $\begin{bmatrix} 4 & -2 & 1 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & 2 & 7 \end{bmatrix}$

4. Each of the above matrices represents a system of equations. For each problem, write the system of equations represented by the original matrix. Determine the solution for each system using the row-reduced matrix you obtained, and then check the solutions in the original system.

5. Solve the following problem by using a matrix to represent the system of equations described in the scenario, and then changing the matrix to row-reduced form to obtain the solution.

   Three of Carlos' and Clarita's friends are purchasing school supplies at the bookstore. Stan buys a notebook, three packages of pencils and two markers for $7.50. Jan buys two notebooks, six packages of pencils and five markers for $15.50. Fran buys a notebook, two packages of pencils and two markers for $6.25. How much do each of these three items cost?

6. Create a linear system that is either dependent (both equations in the system represent the same line) or inconsistent (the equations in the system represent non-intersecting lines). What happens when you try to row reduce the 2×3 matrix that represents this linear system of equations?



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Ready, Set, Go!

Ready

Topic: Solving systems of equations using matrices.

1. In an earlier assignment you worked the following problem:

   "A theater wants to take in $2000 for a certain matinee. Children's tickets cost $5 each and adult tickets cost $10 each. If the theater has a maximum of 350 seats, write a system of equations that can be solved to determine the number of both children and adult tickets the theater can sell."

   Set up a matrix that goes with the situation described above.

Set

Assume that the matrices below represent linear systems of equations. Practice the strategy you used for reducing a given matrix so that the left portion of the matrix (the 2 rows and first 2 columns of entries) has ones on the diagonal. Write a description of what you did to get from one matrix to another in each step of your sequence of matrices.

2. $\begin{bmatrix} 3 & 2 & -6 \\ 1 & 2 & 2 \end{bmatrix}$

3. $\begin{bmatrix} -3 & 1 & -12 \\ 2 & 3 & -15 \end{bmatrix}$

4. $\begin{bmatrix} 7 & 2 & 24 \\ 8 & 2 & 30 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 1 & 9 \\ 10 & -7 & -18 \end{bmatrix}$

Go

Topic: Solving systems of equations

Solve the following systems of equations with a method of your choice.

6. $\begin{cases} x - y = 11 \\ 2x + y = 19 \end{cases}$

7. $\begin{cases} 8x + y = -16 \\ -3x + y = -5 \end{cases}$

8. $\begin{cases} -4x + 9y = 9 \\ x - 3y = -6 \end{cases}$

9. $\begin{cases} -7x = y = -19 \\ -2x + 3y = -19 \end{cases}$

Need help? Check out these related videos:

http://www.khanacademy.org/math/algebra/ck12-algebra-1/v/solving-linear-systems-by-graphing

http://www.khanacademy.org/math/algebra/ck12-algebra-1/v/solving-linear-systems-by-substitution

http://www.khanacademy.org/math/algebra/ck12-algebra-1/v/solving-systems-of-equations-by-elimination

HOMEWORK HELP:
Systems of Equations and Inequalities

Skills students will be working on:

1. Solving systems of equations
   a. special types (consistent, inconsistent, and dependent)
      http://www.youtube.com/watch?v=Ix8Nne-a-KQ
   b. Solve by graphing
   c. Solve by substitution
   d. Solve by elimination
2. Solving systems of two variable linear inequalities
   a. Linear systems of inequalities
   b. Graphing linear inequalities

http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/u06-l3-t1-we3-graphing-systems-of-inequalities

http://www.khanacademy.org/math/algebra/linear-equations-and-inequalitie/v/graphing-inequalities-2

3. Linear equations in standard form
   a. Writing equations in standard form

http://www.khanacademy.org/math/algebra/linear-equations-and-inequalitie/v/linear-equations-in-standard-form

   b. Graphing linear equations in standard form

http://www.khanacademy.org/math/algebra/linear-equations-and-inequalitie/v/converting-to-slope-intercept-form

4. Testing if a point is a solution of the system

http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/testing-a-solution-for-a-system-of-equations

http://www.khanacademy.org/math/algebra/systems-of-eq-and-ineq/v/testing-solutions-for-a-system-of-inequalities

1a. Special Types of Linear Systems

Objectives

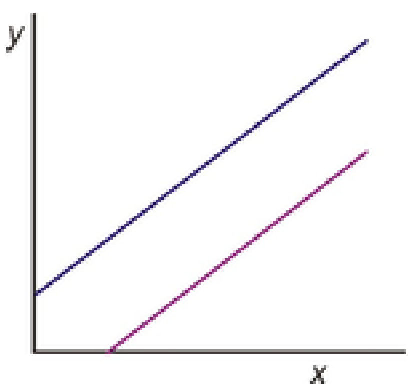
- Identify and understand what is meant by an **inconsistent linear system.**
- Identify and understand what is meant by a **consistent linear system.**
- Identify and understand what is meant by a **dependent linear system.**

Concept

Introduction

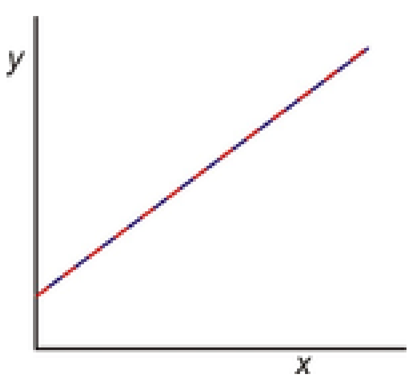
As we saw in Section 7.1, a system of linear equations is a set of linear equations which must be solved together. The lines in the system can be graphed together on the same coordinate graph and the solution to the system is the point at which the two lines intersect.

Or at least that's what usually happens. But what if the lines turn out to be parallel when we graph them?



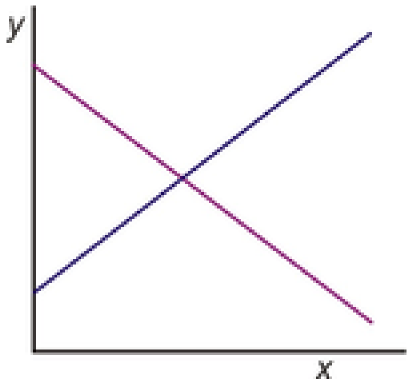
If the lines are parallel, they won't ever intersect. That means that the system of equations they represent has no solution. A system with no solutions is called an **inconsistent system.**

And what if the lines turn out to be identical?



If the two lines are the same, then *every* point on one line is also on the other line, so every point on the line is a solution to the system. The system has an **infinite number** of solutions, and the two equations are really just different forms of the same equation. Such a system is called a **dependent system**.

But usually, two lines cross at exactly one point and the system has exactly one solution:



A system with exactly one solution is called a **consistent system**.

To identify a system as **consistent, inconsistent**, or **dependent**, we can graph the two lines on the same graph and see if they intersect, are parallel, or are the same line. But sometimes it is hard to tell whether two lines are parallel just by looking at a roughly sketched graph.

Another option is to write each line in slope-intercept form and compare the slopes and $y-$ intercepts of the two lines. To do this we must remember that:

- Lines with different slopes always intersect.
- Lines with the same slope but different $y-$intercepts are parallel.
- Lines with the same slope and the same $y-$intercepts are identical.

Example 1

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$2x - 5y = 2$$
$$4x + y = 5$$

Solution

We must rewrite the equations so they are in slope-intercept form

$$\begin{aligned} 2x-5y &= 2 \\ 4x+y &= 5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -5y &= -2x+2 \\ y &= -4x+5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= \tfrac{2}{5}x-\tfrac{2}{5} \\ y &= -4x+5 \end{aligned}$$

The slopes of the two equations are different; therefore the lines must cross at a single point and the system has exactly one solution. This is a **consistent system.**

Example 2

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$3x = 5 - 4y$$
$$6x + 8y = 7$$

Solution

We must rewrite the equations so they are in slope-intercept form

$$\begin{aligned} 3x &= 5 - 4y \\ 6x+8y &= 7 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4y &= -3x + 5 \\ 8y &= -6x+7 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= -\tfrac{3}{4}x + \tfrac{5}{4} \\ y &= -\tfrac{3}{4}x+\tfrac{7}{8} \end{aligned}$$

The slopes of the two equations are the same but the $y-$intercepts are different; therefore the lines are parallel and the system has no solutions. This is an **inconsistent system.**

Example 3

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} x + y &= 3 \\ 3x + 3y &= 9 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$$\begin{aligned} x + y &= 3 & & & y &= -x + 3 & & & y &= -x + 3 \\ & & &\Rightarrow & & & &\Rightarrow & & \\ 3x + 3y &= 9 & & & 3y &= -3x + 9 & & & y &= -x + 3 \end{aligned}$$

The lines are identical; therefore the system has an infinite number of solutions. It is a **dependent system.**

Determining the Type of System Algebraically

A third option for identifying systems as consistent, inconsistent or dependent is to just solve the system and use the result as a guide.

Example 4

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned} 10x - 3y &= 3 \\ 2x + y &= 9 \end{aligned}$$

Solution

Let's solve this system using the substitution method.

Solve the second equation for y:

$$2x + y = 9 \Rightarrow y = -2x + 9$$

Substitute that expression for y in the first equation:

$$\begin{aligned} 10x - 3y &= 3 \\ 10x - 3(-2x + 9) &= 3 \\ 10x + 6x - 27 &= 3 \\ 16x &= 30 \\ x &= \frac{15}{8} \end{aligned}$$

Substitute the value of x back into the second equation and solve for y:

$$2x + y = 9 \Rightarrow y = -2x + 9 \Rightarrow y = -2 \cdot \frac{15}{8} + 9 \Rightarrow y = \frac{21}{4}$$

The solution to the system is $\left(\frac{15}{8}, \frac{21}{4}\right)$. The system is **consistent** since it has only one solution.

Example 5

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$3x - 2y = 4$$
$$9x - 6y = 1$$

Solution

Let's solve this system by the method of multiplication.

Multiply the first equation by 3:

$$\begin{array}{l} 3(3x - 2y = 4) \\ 9x - 6y = 1 \end{array} \quad \Rightarrow \quad \begin{array}{l} 9x - 6y = 12 \\ 9x - 6y = 1 \end{array}$$

Add the two equations:

$$\begin{array}{r} 9x - 6y = 4 \\ \underline{9x - 6y = 1} \\ 0 = 13 \quad \text{This statement is not true.} \end{array}$$

If our solution to a system turns out to be a statement that is not true, then the system doesn't really have a solution; it is **inconsistent.**

Example 6

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$4x + y = 3$$
$$12x + 3y = 9$$

Solution

Let's solve this system by substitution.

Solve the first equation for y:

$4x + y = 3 \Rightarrow y = -4x + 3$

Substitute this expression for y in the second equation:

$$\begin{aligned} 12x + 3y &= 9 \\ 12x + 3(-4x + 3) &= 9 \\ 12x - 12x + 9 &= 9 \\ 9 &= 9 \end{aligned}$$

This statement is always true.

If our solution to a system turns out to be a statement that is always true, then the system is **dependent.**

A second glance at the system in this example reveals that the second equation is three times the first equation, so the two lines are identical. The system has an infinite number of solutions because they are really the same equation and trace out the same line.

Let's clarify this statement. An infinite number of solutions does not mean that *any* ordered pair (x, y) satisfies the system of equations. Only ordered pairs that solve the equation in the system (either one of the equations) are also solutions to the system. There are infinitely many of these solutions to the system because there are infinitely many points on any one line.

For example, (1, -1) is a solution to the system in this example, and so is (-1, 7). Each of them fits both the equations because both equations are really the same equation. But (3, 5) doesn't fit either equation and is not a solution to the system.

In fact, for every $x-$value there is just one $y-$value that fits both equations, and for every $y-$value there is exactly one $x-$value—just as there is for a single line.

Let's summarize how to determine the type of system we are dealing with algebraically.

1. A **consistent system** will always give exactly one solution.
2. An **inconsistent system** will yield a statement that is *always false* (like $0 = 13$).
3. A **dependent system** will yield a statement that is *always true* (like $9 = 9$).

Applications

In this section, we'll see how consistent, inconsistent and dependent systems might arise in real life.

Example 7

The movie rental store CineStar offers customers two choices. Customers can pay a yearly membership of $45 and then rent each movie for $2 or they can choose not to pay the membership fee and rent each movie for $3.50. How many movies would you have to rent before the membership becomes the cheaper option?

Solution

Let's translate this problem into algebra. Since there are two different options to consider, we can write two different equations and form a system.

The choices are "membership" and "no membership." We'll call the number of movies you rent x and the total cost of renting movies for a year y.

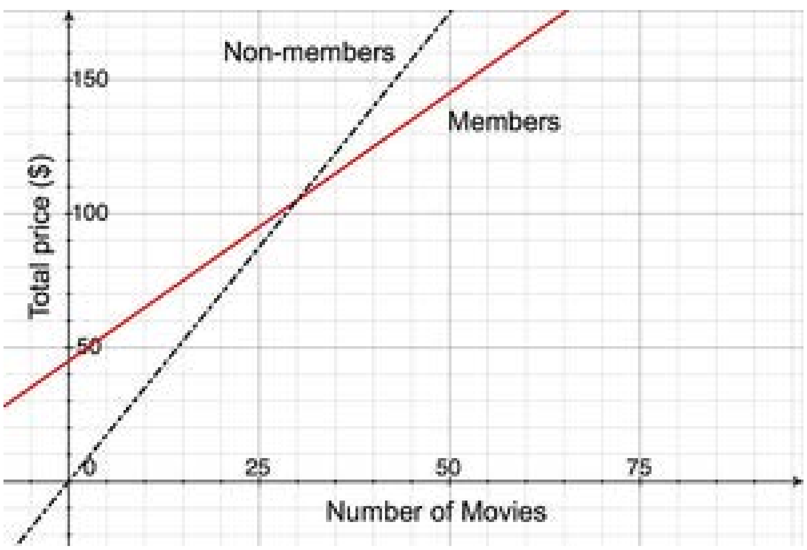
	flat fee	rental fee	total
membership	$45	$2x$	$y = 45 + 2x$
no membership	$0	$3.50x$	$y = 3.5x$

The flat fee is the dollar amount you pay per year and the rental fee is the dollar amount you pay when you rent a movie. For the membership option the rental fee is $2x$, since you would pay $2 for each movie you rented; for the no membership option the rental fee is $3.50x$, since you would pay $3.50 for each movie you rented.

Our system of equations is:

$$\begin{aligned} y &= 45 + 2x \\ y &= 3.50x \end{aligned}$$

Here's a graph of the system:



Now we need to find the exact intersection point. Since each equation is already solved for y, we can easily solve the system with substitution. Substitute the second equation into the first one:

$$\begin{aligned} y &= 45 + 2x \\ y &= 3.50x \end{aligned} \Rightarrow 3.50x = 45 + 2x \Rightarrow 1.50x = 45 \Rightarrow x = 30 \text{ movies}$$

You would have to rent **30 movies per year** before the membership becomes the better option.

This example shows a real situation where a consistent system of equations is useful in finding a solution. Remember that for a consistent system, the lines that make up the system intersect at single point. In other words, the lines are not parallel or the slopes are different.

In this case, the slopes of the lines represent the price of a rental per movie. The lines cross because the price of rental per movie is different for the two options in the problem

Now let's look at a situation where the system is inconsistent. From the previous explanation, we can conclude that the lines will not intersect if the slopes are the same (and the $y-$intercept is different). Let's change the previous problem so that this is the case.

Example 8

Two movie rental stores are in competition. Movie House charges an annual membership of $30 and charges $3 per movie rental. Flicks for Cheap charges an annual membership of $15 and charges $3 per movie rental. After how many movie rentals would Movie House become the better option?

Solution

It should already be clear to see that Movie House will never become the better option, since its membership is more expensive and it charges the same amount per movie as Flicks for Cheap.

The lines on a graph that describe each option have different $y-$intercepts—namely 30 for Movie House and 15 for Flicks for Cheap—but the same slope: 3 dollars per movie. This means that the lines are parallel and so the system is inconsistent.

Now let's see how this works algebraically. Once again, we'll call the number of movies you rent x and the total cost of renting movies for a year y.

	flat fee	**rental fee**	**total**
Movie House	$30	$3x$	$y = 30 + 3x$

Flicks for Cheap	\$15	$3x$	$y = 15 + 3x$

The system of equations that describes this problem is:

$$y = 30 + 3x$$
$$y = 15 + 3x$$

Let's solve this system by substituting the second equation into the first equation:

$$\begin{array}{l} y = 30 + 3x \\ y = 15 + 3x \end{array} \Rightarrow 15+3x = 30+3x \Rightarrow 15 = 30 \qquad \text{This statement is always false.}$$

This means that the system is **inconsistent.**

Example 9

Peter buys two apples and three bananas for \$4. Nadia buys four apples and six bananas for \$8 from the same store. How much does one banana and one apple costs?

Solution

We must write two equations: one for Peter's purchase and one for Nadia's purchase.

Let's say a is the cost of one apple and b is the cost of one banana.

	cost of apples	**cost of bananas**	**total cost**
Peter	$2a$	$3b$	$2a + 3b = 4$

Nadia	$4a$	$6b$	$4a + 6b = 8$

The system of equations that describes this problem is:

$$\begin{aligned} 2a + 3b &= 4 \\ 4a + 6b &= 8 \end{aligned}$$

Let's solve this system by multiplying the first equation by -2 and adding the two equations:

$$\begin{array}{lcr} -2(2a + 3b = 4) & & -4a - 6b = -8 \\ & \Rightarrow & \\ 4a + 6b = 8 & & \underline{4a + 6b = 8} \\ & & 0 + 0 = 0 \end{array}$$

This statement is always true. This means that the system is **dependent.**

Looking at the problem again, we can see that we were given exactly the same information in both statements. If Peter buys two apples and three bananas for $4, it makes sense that if Nadia buys twice as many apples (four apples) and twice as many bananas (six bananas) she will pay twice the price ($8). Since the second equation doesn't give us any new information, it doesn't make it possible to find out the price of each fruit.

1b. Linear Systems by Graphing

Objectives

- Determine whether an ordered pair is a solution to a system of equations.
- http://www.youtube.com/watch?v=ssFyt4_htOw
- Solve a system of equations graphically.
- Solve a system of equations graphically with a graphing calculator.
- Solve word problems using systems of equations.

Concept

Introduction

In this lesson, we'll discover methods to determine if an ordered pair is a solution to a system of two equations. Then we'll learn to solve the two equations graphically and confirm that the solution is the point where the two lines intersect. Finally, we'll look at real-world problems that can be solved using the methods described in this chapter.

Determine Whether an Ordered Pair is a Solution to a System of Equations

A linear system of equations is a set of equations that must be solved together to find the one solution that fits them both.

Consider this system of equations:

$$y = x + 2$$
$$y = -2x + 1$$

Since the two lines are in a system, we deal with them together by graphing them on the same coordinate axes. We can use any method to graph them; let's do it by making a table of values for each line.

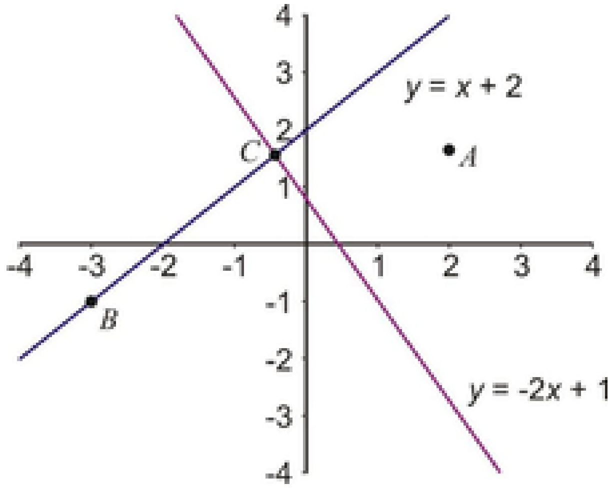
Line $y = x + 2$ 1:

x	y
0	2
1	3

Line $y = -2x + 1$ 2:

x	y

0	1
1	-1



We already know that any point that lies on a line is a solution to the equation for that line. That means that any point that lies on *both* lines in a system is a solution to both equations.

So in this system:

- Point A is not a solution to the system because it does not lie on either of the lines.
- Point B is not a solution to the system because it lies only on the blue line but not on the red line.
- Point C is a solution to the system because it lies on both lines at the same time.

In fact, point C is the only solution to the system, because it is the only point that lies on both lines. For a system of equations, the geometrical solution is the intersection of the two lines in the system. The algebraic solution is the ordered pair that solves both equations—in other words, the coordinates of that intersection point.

You can confirm the solution by plugging it into the system of equations, and checking that the solution works in each equation.

Example 1

Determine which of the points (1, 3), (0, 2), or (2, 7) is a solution to the following system of equations:

$$y = 4x - 1$$
$$y = 2x + 3$$

Solution

To check if a coordinate point is a solution to the system of equations, we plug each of the x and y values into the equations to see if they work.

Point (1, 3):

$$\begin{aligned} y &= 4x - 1 \\ 3 &\overset{?}{=} 4(1) - 1 \\ 3 &= 3 \text{ solution checks} \end{aligned}$$

$$\begin{aligned} y &= 2x + 3 \\ 3 &\overset{?}{=} 2(1) + 3 \\ 3 &\neq 5 \text{ solution does not check} \end{aligned}$$

Point (1, 3) is on the line $y = 4x - 1$, but it is not on the line $y = 2x + 3$, so it is not a solution to the system.

Point (0, 2):

$$\begin{aligned} y &= 4x - 1 \\ 2 &\overset{?}{=} 4(0) - 1 \\ 2 &\neq -1 \text{ solution does not check} \end{aligned}$$

Point (0, 2) is not on the line $y = 4x - 1$, so it is not a solution to the system. Note that it is not necessary to check the second equation because the point needs to be on both lines for it to be a solution to the system.

Point (2, 7):

$$\begin{aligned} y &= 4x - 1 \\ 7 \stackrel{?}{=} &\ 4(2) - 1 \\ 7 &= 7 \text{ solution checks} \end{aligned}$$

$$\begin{aligned} y &= 2x + 3 \\ 7 \stackrel{?}{=} &\ 2(2) + 3 \\ 7 &= 7 \text{ solution checks} \end{aligned}$$

Point (2, 7) is a solution to the system since it lies on both lines.

The solution to the system is the point (2, 7).

Determine the Solution to a Linear System by Graphing

The solution to a linear system of equations is the point, (if there is one) that lies on both lines. In other words, the solution is the point where the two lines intersect.

We can solve a system of equations by graphing the lines on the same coordinate plane and reading the intersection point from the graph.

This method most often offers only approximate solutions, so it's not sufficient when you need an exact answer. However, graphing the system of equations can be a good way to get a sense of what's really going on in the problem you're trying to solve, especially when it's a real-world problem.

Example 2

Solve the following system of equations by graphing:

$$\begin{aligned} y &= 3x - 5 \\ y &= -2x + 5 \end{aligned}$$

Solution

Graph both lines on the same coordinate axis using any method you like.

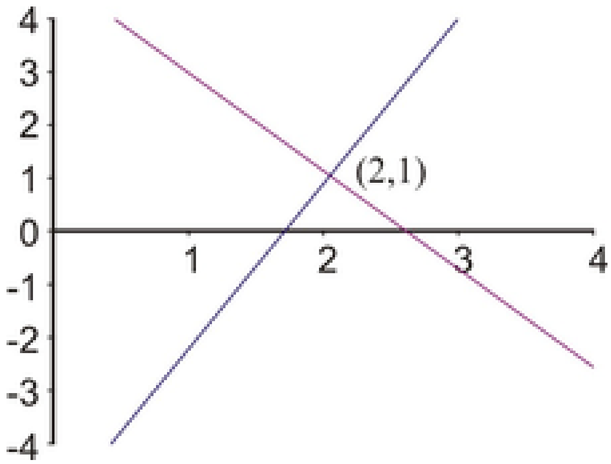
In this case, let’s make a table of values for each line.

Line 1: $y = 3x - 5$

x	y
1	-2
2	1

Line 2: $y = -2x + 5$

x	y
1	3
2	1



The solution to the system is given by the intersection point of the two lines. The graph shows that the lines intersect at point (2, 1). So **the solution is** $x = 2, y = 1$ **or (2, 1).**

Example 3

Solve the following system of equations by graphing:

$$2x + 3y = 6$$
$$4x - y = -2$$

Solution

Since the equations are in standard form, this time we'll graph them by finding the $x-$ and $y-$intercepts of each of the lines.

Line $2x + 3y = 6$ **1:**

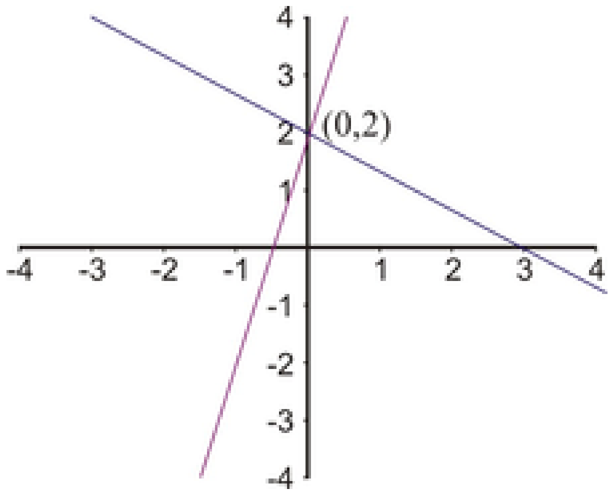
$x-$intercept: set $y = 0 \Rightarrow 2x = 6 \Rightarrow x = 3$ so the intercept is (3, 0)

$y-$intercept: set $x = 0 \Rightarrow 3y = 6 \Rightarrow y = 2$ so the intercept is (0, 2)

Line 2: $-4x + y = 2$

$x-$intercept: set $y = 0 \Rightarrow -4x = 2 \Rightarrow x = -\frac{1}{2}$ so the intercept $\left(-\frac{1}{2}, 0\right)$ is

$y-$intercept: set $x = 0 \Rightarrow y = 2$ so the intercept is (0, 2)



The graph shows that the lines intersect at (0, 2). Therefore, **the solution to the system of equations is** $x = 0, y = 2$.

Solving a System of Equations Using a Graphing Calculator

As an alternative to graphing by hand, you can use a graphing calculator to find or check solutions to a system of equations.

Example 4

Solve the following system of equations using a graphing calculator.

$$\begin{aligned} x - 3y &= 4 \\ 2x + 5y &= 8 \end{aligned}$$

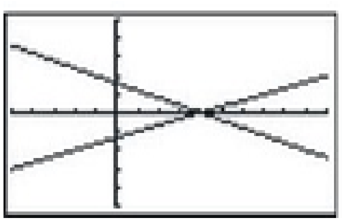
To input the equations into the calculator, you need to rewrite them in slope-intercept form (that is, $y = mx + b$ form).

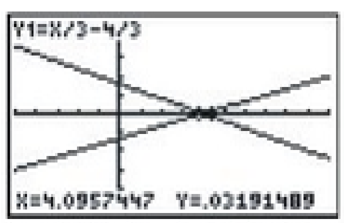
$$\begin{aligned} x - 3y &= 4 \\ 2x + 5y &= 8 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= \tfrac{1}{3}x - \tfrac{4}{3} \\ y &= -\tfrac{2}{5}x + \tfrac{8}{5} \end{aligned}$$

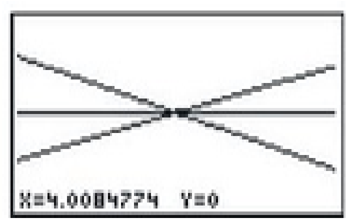
Press the **[y=]** button on the graphing calculator and enter the two functions as:

$$Y_1 = \frac{x}{3} - \frac{4}{3}$$
$$Y_2 = \frac{-2x}{5} + \frac{8}{5}$$

Now press **[GRAPH]**. Here's what the graph should look like on a TI-83 family graphing calculator with the window set to $-5 \le x \le 10$ and $-5 \le y \le 5$.







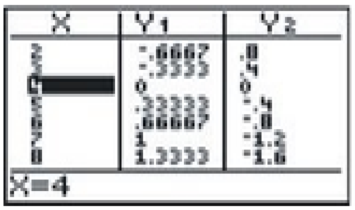
There are a few different ways to find the intersection point.

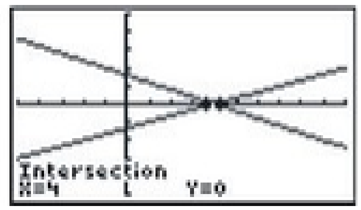
Option 1: Use **[TRACE]** and move the cursor with the arrows until it is on top of the intersection point. The values of the coordinate point will be shown on the bottom of the screen. The second screen above shows the values to be $X = 4.0957447$ and $Y = 0.03191489$.

Use the **[ZOOM]** function to zoom into the intersection point and find a more accurate result. The third screen above shows the system of equations after zooming in several times. A more accurate solution appears to be $X = 4$ and $Y = 0$.

Option 2 Look at the table of values by pressing **[2nd] [GRAPH]**. The first screen below shows a table of values for this system of equations. Scroll down until the $Y-$values for the two functions are the same. In this case this occurs at $X = 4$ and $Y = 0$.

(Use the **[TBLSET]** function to change the starting value for your table of values so that it is close to the intersection point and you don't have to scroll too long. You can also improve the accuracy of the solution by setting the value of Δ Table smaller.)





Option 3 Using the **[2nd] [TRACE]** function gives the second screen shown above.

Scroll down and select "intersect."

The calculator will display the graph with the question **[FIRSTCURVE]**? Move the cursor along the first curve until it is close to the intersection and press **[ENTER]**.

The calculator now shows **[SECONDCURVE]**?

Move the cursor to the second line (if necessary) and press **[ENTER]**.

The calculator displays **[GUESS]**?

Press **[ENTER]** and the calculator displays the solution at the bottom of the screen (see the third screen above).

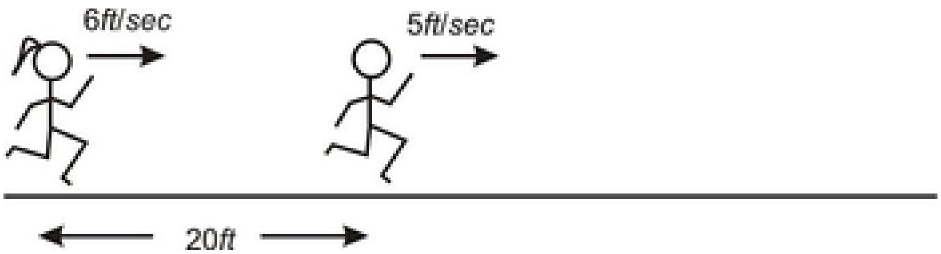
The point of intersection is $X = 4$ and $Y = 0$. Note that with this method, the calculator works out the intersection point for you, which is generally more accurate than your own visual estimate.

Solve Real-World Problems Using Graphs of Linear Systems

Consider the following problem:

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

Let's start by drawing a sketch. Here's what the race looks like when Nadia starts running; we'll call this time $t = 0$.



Now let's define two variables in this problem:

$t =$ the time from when Nadia starts running

$d =$ the distance of the runners from the starting point.

Since there are two runners, we need to write equations for each of them. That will be the *system of equations* for this problem.

For each equation, we use the $\text{distance} = \text{speed} \times \text{time}$ formula:

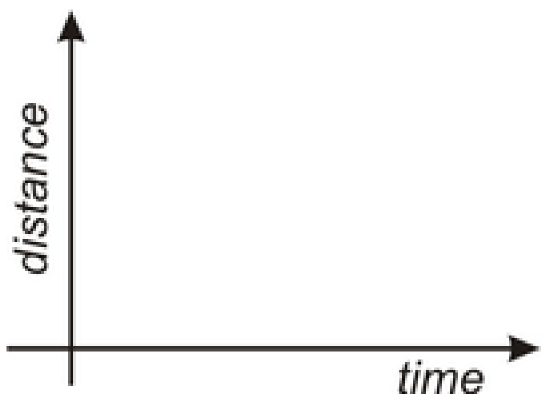
Nadia's $d = 6t$ equation:

Peter's $d = 5t + 20$ equation:

(Remember that Peter was already 20 feet from the starting point when Nadia started running.)

Let's graph these two equations on the same coordinate axes.

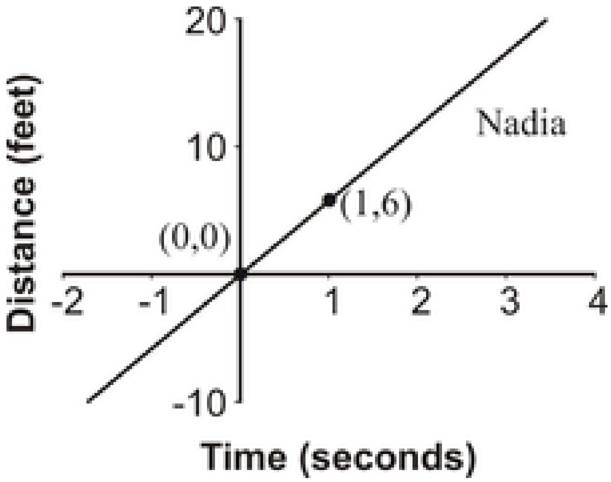
Time should be on the horizontal axis since it is the independent variable. Distance should be on the vertical axis since it is the dependent variable.



We can use any method for graphing the lines, but in this case we'll use the *slope–intercept* method since it makes more sense physically.

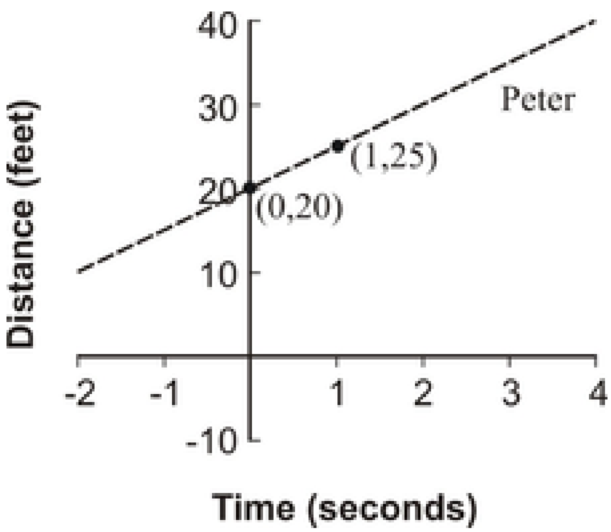
To graph the line that describes Nadia's run, start by graphing the $y-$intercept: (0, 0). (If you don't see that this is the $y-$intercept, try plugging in the test-value of $x = 0$.)

The slope tells us that Nadia runs 6 feet every one second, so another point on the line is (1, 6). Connecting these points gives us Nadia's line:

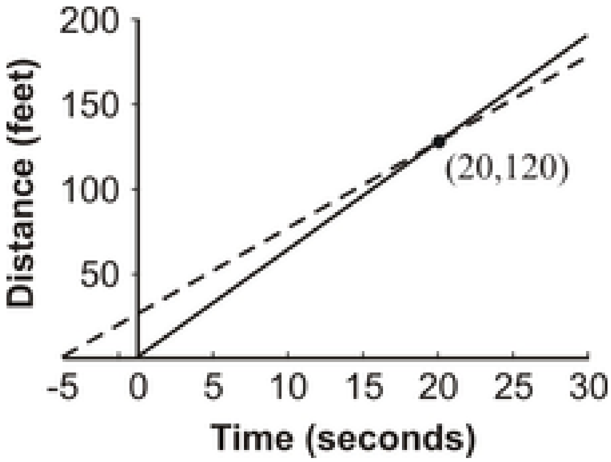


To graph the line that describes Peter's run, again start with the $y-$intercept. In this case this is the point (0, 20).

The slope tells us that Peter runs 5 feet every one second, so another point on the line is (1, 25). Connecting these points gives us Peter's line:



In order to find when and where Nadia and Peter meet, we'll graph both lines on the same graph and extend the lines until they cross. The crossing point is the solution to this problem.



The graph shows that Nadia and Peter meet **20 seconds after Nadia starts running, and 120 feet from the starting point.**

These examples are great at demonstrating that the solution to a system of linear equations means the point at which the lines intersect. This is, in fact, the greatest strength of the graphing method because it offers a very visual representation of system of equations and its solution. You can also see, though, that finding the solution from a graph requires very careful graphing of the lines, and is really only practical when you're sure that the solution gives integer values for x and y. Usually, this method can only offer approximate solutions to systems of equations, so we need to use other methods to get an exact solution.

1c. Solving Linear Systems by Substitution

Objectives

- Solve systems of equations with two variables by substituting for either variable.
- Manipulate standard form equations to isolate a single variable.
- Solve real-world problems using systems of equations.
- Solve mixture problems using systems of equations.

Concept

Introduction

In this lesson, we'll learn to solve a system of two equations using the method of substitution.

Solving Linear Systems Using Substitution of Variable Expressions

Let's look again at the problem about Peter and Nadia racing.

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

In that example we came up with two equations:

Nadia's $d = 6t$ equation:

Peter's $d = 5t + 20$ equation:

Each equation produced its own line on a graph, and to solve the system we found the point at which the lines intersected—the point where the values for d and t satisfied **both** relationships. When the values for d and t are equal, that means that Peter and Nadia are at the same place at the same time.

But there's a faster way than graphing to solve this system of equations. Since we want the value of d to be the same in both equations, we could just set the two right-hand sides of the equations equal to each other to solve for t. That is, if $d = 6t$ and $d = 5t + 20$, and the two d's are equal to each other, then by the transitive property we have $6t = 5t + 20$. We can solve this for t:

$$\begin{aligned} 6t &= 5t + 20 && \mathit{subtract\ 5t\ from\ both\ sides}: \\ t &= 20 && \mathit{substitute\ this\ value\ for\ t\ into\ Nadias\ equation}: \\ d &= 6 \cdot 20 = 120 \end{aligned}$$

Even if the equations weren't so obvious, we could use simple algebraic manipulation to find an expression for one variable in terms of the other. If we rearrange Peter's equation to isolate t:

$$\begin{aligned}
d &= 5t + 20 && \textit{subtract 20 from both sides:} \\
d - 20 &= 5t && \textit{divide by 5:} \\
\frac{d-20}{5} &= t
\end{aligned}$$

We can now ***substitute*** this expression for t into Nadia's equation $(d = 6t)$ to solve:

$$\begin{aligned}
d &= 6\left(\frac{d-20}{5}\right) && \textit{multiply both sides by 5:} \\
5d &= 6(d-20) && \textit{distribute the 6:} \\
5d &= 6d - 120 && \textit{subtract 6d from both sides:} \\
-d &= -120 && \textit{divide by } -1: \\
d &= 120 && \textit{substitute value for d into our expression for t:} \\
t &= \frac{120-20}{5} = \frac{100}{5} = 20
\end{aligned}$$

So we find that Nadia and Peter meet 20 seconds after they start racing, at a distance of 120 feet away.

The method we just used is called the **Substitution Method.** In this lesson you'll learn several techniques for isolating variables in a system of equations, and for using those expressions to solve systems of equations that describe situations like this one.

Example 1

Let's look at an example where the equations are written in **standard form.**

Solve the system

$$\begin{aligned}
2x + 3y &= 6 \\
-4x + y &= 2
\end{aligned}$$

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of y is 1. So the easiest way to start is to use this equation to solve for y.

Solve the second equation for y:

$$\begin{aligned} -4x + y &= 2 && \textit{add } 4x \textit{ to both sides}: \\ y &= 2 + 4x \end{aligned}$$

Substitute this expression into the first equation:

$$\begin{aligned} 2x + 3(2 + 4x) &= 6 && \textit{distribute the } 3: \\ 2x + 6 + 12x &= 6 && \textit{collect like terms}: \\ 14x + 6 &= 6 && \textit{subtract } 6 \textit{ from both sides}: \\ 14x &= 0 && \textit{and hence}: \\ x &= 0 \end{aligned}$$

Substitute back into our expression for y:

$$y = 2 + 4 \cdot 0 = 2$$

As you can see, we end up with the same solution $(x = 0, y = 2)$ that we found when we graphed these functions back in Lesson 7.1. So long as you are careful with the algebra, the substitution method can be a very efficient way to solve systems.

Next, let's look at a more complicated example. Here, the values of x and y we end up with aren't whole numbers, so they would be difficult to read off a graph!

Example 2

Solve the system

$$2x + 3y = 3$$
$$2x - 3y = -1$$

Again, we start by looking to isolate one variable in either equation. In this case it doesn't matter which equation we use—all the variables look about equally easy to solve for.

So let's solve the first equation for x:

$$\begin{aligned} 2x + 3y &= 3 && \textit{subtract } 3y \textit{ from both sides}: \\ 2x &= 3 - 3y && \textit{divide both sides by } 2: \\ x &= \frac{1}{2}(3 - 3y) \end{aligned}$$

Substitute this expression into the second equation:

$$\begin{aligned} \cancel{2} \cdot \frac{1}{\cancel{2}}(3 - 3y) - 3y &= -1 && \textit{cancel the fraction and re-write terms}: \\ 3 - 3y - 3y &= -1 && \textit{collect like terms}: \\ 3 - 6y &= -1 && \textit{subtract } 3 \textit{ from both sides}: \\ -6y &= -4 && \textit{divide by } -6: \\ y &= \frac{2}{3} \end{aligned}$$

Substitute into the expression we got for x:

$$x = \frac{1}{2}\left(3 - \cancel{3}\left(\frac{2}{\cancel{3}}\right)\right)$$
$$x = \frac{1}{2}$$

So our solution is $x = \frac{1}{2}, y = \frac{2}{3}$. You can see how the graphical solution $\left(\frac{1}{2}, \frac{2}{3}\right)$ might have been difficult to read accurately off a graph!

Solving Real-World Problems Using Linear Systems

Simultaneous equations can help us solve many real-world problems. We may be considering a purchase—for example, trying to decide whether it's cheaper to buy an item online where you pay shipping or at the store where you do not. Or you may wish to join a CD music club, but aren't sure if you would really save any money by buying a new CD every month in that way. Or you might be considering two different phone contracts. Let's look at an example of that now.

Example 3

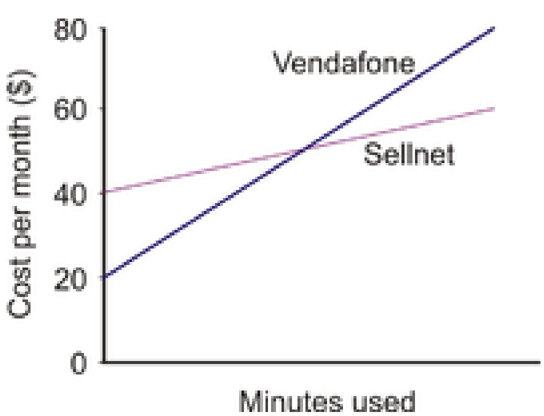
Anne is trying to choose between two phone plans. The first plan, with Vendafone, costs $20 per month, with calls costing an additional 25 cents per minute. The second company, Sellnet, charges $40 per month, but calls cost only 8 cents per minute. Which should she choose?

You should see that Anne's choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations for the cost in dollars in terms of the minutes used. Since the *number of minutes* is the independent variable, it will be our x. Cost is *dependent* on minutes – the *cost per month* is the dependent variable and will be assigned y.

For $y = 0.25x + 20$ *Vendafone:*

For $y = 0.08x + 40$ *Sellnet:*

By writing the equations in slope-intercept form $(y = mx + b)$, you can sketch a graph to visualize the situation:



The line for Vendafone has an intercept of 20 and a slope of 0.25. The Sellnet line has an intercept of 40 and a slope of 0.08 (which is roughly a third of the Vendafone line's slope). In order to help Anne decide which to choose, we'll find where the two lines cross, by solving the two equations as a system.

Since equation 1 gives us an expression for $y(0.25x + 20)$, we can substitute this expression directly into equation 2:

$$\begin{aligned} 0.25x + 20 &= 0.08x + 40 && \textit{subtract 20 from both sides:} \\ 0.25x &= 0.08x + 20 && \textit{subtract } 0.08x \textit{ from both sides:} \\ 0.17x &= 20 && \textit{divide both sides by } 0.17: \\ x &= 117.65 \textit{ minutes} && \textit{rounded to 2 decimal places.} \end{aligned}$$

So if Anne uses 117.65 minutes a month (although she can't really do *exactly* that, because phone plans only count whole numbers of minutes), the phone plans will cost the same. Now we need to look at the graph to see which plan is better if she uses more minutes than that, and which plan is better if she uses fewer. You can see that the Vendafone plan costs more when she uses more minutes, and the Sellnet plan costs more with fewer minutes.

So, **if Anne will use 117 minutes or less every month she should choose *Vendafone.* If she plans on using 118 or more minutes she should choose *Sellnet.***

Mixture Problems

Systems of equations crop up frequently in problems that deal with mixtures of two things—chemicals in a solution, nuts and raisins, or even the change in your pocket! Let's look at some examples of these.

Example 4

Janine empties her purse and finds that it contains only nickels (worth 5 cents each) and dimes (worth 10 cents each). If she has a total of 7 coins and they have a combined value of 45 cents, how many of each coin does she have?

Since we have 2 types of coins, let's call the number of nickels x and the number of dimes y. We are given two key pieces of information to make our equations: the number of coins and their value.

$$\begin{array}{lll} \#\text{ of coins equation:} & x+y=7 & (number\ of\ nickels)+(number\ of\ dimes) \\ \text{value equation:} & 5x+10y=55 & (since\ nickels\ are\ worth\ 5c\ and\ dimes\ 10c) \end{array}$$

We can quickly rearrange the first equation to isolate x:

$$\begin{array}{ll} x=7-y & now\ substitute\ into\ equation\ 2: \\ 5(7-y)+10y=55 & distribute\ the\ 5: \\ 35-5y+10y=55 & collect\ like\ terms: \\ 35+5y=55 & subtract\ 35\ from\ both\ sides: \\ 5y=20 & divide\ by\ 5: \\ \underline{y=4} & substitute\ back\ into\ equation\ 1: \\ x+4=7 & subtract\ 4\ from\ both\ sides: \\ \underline{x=3} & \end{array}$$

Janine has 3 nickels and 4 dimes.

Sometimes a question asks you to determine (from concentrations) how much of a particular substance to use. The substance in question could be something like coins as above, or it could be a chemical in solution, or even heat. In such a case, you need to know the amount of whatever substance is in each part. There are several common situations where to get one equation you simply add two given quantities, but to get the second equation you need to use a **product.** Three examples are below.

Type of mixture	**First equation**	**Second equation**
Coins (items with $ value)	total number of items (n_1+n_2)	total ***value*** (item value $\times$ no. of items)
Chemical solutions	total solution volume (V_1+V_2)	amount of ***solute*** (vol $\times$ concentration)
Density of two substances	total amount or volume of mix	total ***mass*** (volume $\times$ density)

For example, when considering mixing chemical solutions, we will most likely need to consider the total amount of *solute* in the individual parts and in the final mixture. (A solute is the chemical that is dissolved in a solution. An example of a solute is salt when added to water to make a brine.) To find the total amount, simply multiply the amount of the mixture by the **fractional concentration**. To illustrate, let's look at an example where you are given amounts relative to the whole.

Example 5

A chemist needs to prepare 500 ml of copper-sulfate solution with a 15% concentration. She wishes to use a high concentration solution (60%) and dilute it with a low concentration solution (5%) in order to do this. How much of each solution should she use?

Solution

To set this problem up, we first need to define our variables. Our unknowns are the amount of concentrated solution (x) and the amount of dilute solution (y). We will also convert the percentages (60%, 15% and 5%) into decimals (0.6, 0.15 and 0.05). The two pieces of critical information are the final volume (500 ml) and the final amount of solute (15% of $500\ ml = 75\ ml$). Our equations will look like this:

Volume $x + y = 500$ equation:

Solute $0.6x + 0.05y = 75$ equation:

To isolate a variable for substitution, we can see it's easier to start with equation 1:

$$\begin{aligned}
&x + y = 500 && \textit{subtract } y \textit{ from both sides}: \\
&x = 500 - y && \textit{now substitute into equation } 2: \\
&0.6(500 - y) + 0.05y = 75 && \textit{distribute the } 0.6: \\
&300 - 0.6y + 0.05y = 75 && \textit{collect like terms}: \\
&300 - 0.55y = 75 && \textit{subtract } 300 \textit{ from both sides}: \\
&-0.55y = -225 && \textit{divide both sides by } -0.55: \\
&\underline{y = 409\ ml} && \textit{substitute back into equation for } x: \\
&x = 500 - 409 = \underline{91\ ml} &&
\end{aligned}$$

So the chemist should mix 91 ml of the 60% solution with 409 ml of the 5% solution.

Further Practice

For lots more practice solving linear systems, check out this web page: http://www.algebra.com/algebra/homework/coordinate/practice-linear-system.epl

After clicking to see the solution to a problem, you can click the back button and then click Try Another Practice Linear System to see another problem.

1d. Solving Linear Systems by Elimination

Objectives

- Solve a linear system of equations using elimination by addition.
- Solve a linear system of equations using elimination by subtraction.
- Solve a linear system of equations by multiplication and then addition or subtraction.
- Compare methods for solving linear systems.
- Solve real-world problems using linear systems by any method.

Concept

Introduction

In this lesson, we'll see how to use simple addition and subtraction to simplify our system of equations to a single equation involving a single variable. Because we go from two unknowns (x and y) to a single unknown (either x or y), this method is often referred to by ***solving by elimination***. We eliminate one variable in order to make our equations solvable! To illustrate this idea, let's look at the simple example of buying apples and bananas.

Example 1

If one apple plus one banana costs $1.25 and one apple plus 2 bananas costs $2.00, how much does one banana cost? One apple?

It shouldn't take too long to discover that each banana costs $0.75. After all, the second purchase just contains 1 more banana than the first, and costs $0.75 more, so that one banana must cost $0.75.

Here’s what we get when we describe this situation with algebra:

$$\begin{aligned} a + b &= 1.25 \\ a + 2b &= 2.00 \end{aligned}$$

Now we can subtract the number of apples and bananas in the first equation from the number in the second equation, and also subtract the cost in the first equation from the cost in the second equation, to get the *difference* in cost that corresponds to the *difference* in items purchased.

$$(a + 2b) - (a + b) = 2.00 - 1.25 \rightarrow b = 0.75$$

That gives us the cost of one banana. To find out how much one apple costs, we subtract $0.75 from the total cost of one apple and one banana.

$$a + 0.75 = 1.25 \rightarrow a = 1.25 - 0.75 \rightarrow a = 0.50$$

So an apple costs 50 cents.

To solve systems using addition and subtraction, we’ll be using exactly this idea – by looking at the ***sum*** or ***difference*** of the two equations we can determine a value for one of the unknowns.

Solving Linear Systems Using Addition of Equations

Often considered the easiest and most powerful method of solving systems of equations, the addition (or elimination) method lets us combine two equations in such a way that the resulting equation has only one variable. We can then use simple algebra to solve for that variable. Then, if we need to, we can substitute the value we get for that variable back into either one of the original equations to solve for the other variable.

Example 2

Solve this system by addition:

$$3x + 2y = 11$$
$$5x - 2y = 13$$

Solution

We will add **everything** on the left of the equals sign from both equations, and this will be equal to the sum of everything on the right:

$$(3x + 2y) + (5x - 2y) = 11 + 13 \rightarrow 8x = 24 \rightarrow x = 3$$

A simpler way to visualize this is to keep the equations as they appear above, and to add them together vertically, going down the columns. However, just like when you add units, tens and hundreds, you MUST be sure to keep the xs and ys in their own columns. You may also wish to use terms like $0y$ as a placeholder!

$$\begin{array}{rl} & 3x + 2y = 11 \\ + & (5x - 2y) = 13 \\ \hline & 8x + 0y = 24 \end{array}$$

Again we get $8x = 24$, or $x = 3$. To find a value for y, we simply substitute our value for x back in.

Substitute $x = 3$ into the second equation:

$$\begin{aligned} 5 \cdot 3 - 2y &= 13 \qquad since\ 5 \times 3 = 15,\ we\ subtract\ 15\ from\ both\ sides: \\ -2y &= -2 \qquad divide\ by\ -2\ to\ get: \\ y &= 1 \end{aligned}$$

The reason this method worked is that the $y-$coefficients of the two equations were opposites of each other: 2 and -2. Because they were opposites, they canceled each other out when we added the two equations together, so our final equation had no $y-$term in it and we could just solve it for x.

In a little while we'll see how to use the addition method when the coefficients are not opposites, but for now let's look at another example where they are.

Example 3

Andrew is paddling his canoe down a fast-moving river. Paddling downstream he travels at 7 miles per hour, relative to the river bank. Paddling upstream, he moves slower, traveling at 1.5 miles per hour. If he paddles equally hard in both directions, how fast is the current? How fast would Andrew travel in calm water?

Solution

First we convert our problem into equations. We have two unknowns to solve for, so we'll call the speed that Andrew paddles at x, and the speed of the river y. When traveling downstream, Andrew speed is boosted by the river current, so his total speed is his paddling speed *plus* the speed of the river $(x + y)$. Traveling upstream, the river is working against him, so his total speed is his paddling speed *minus* the speed of the river $(x - y)$.

Downstream $x + y = 7$ Equation:

Upstream $x - y = 1.5$ Equation:

Next we'll eliminate one of the variables. If you look at the two equations, you can see that the coefficient of y is +1 in the first equation and -1 in the second. Clearly $(+1) + (-1) = 0$, so this is the variable we will eliminate. To do this we simply add equation 1 to equation 2. We must be careful to collect like terms, and make sure that everything on the left of the equals sign stays on the left, and everything on the right stays on the right:

$$(x + y) + (x - y) = 7 + 1.5 \Rightarrow 2x = 8.5 \Rightarrow x = 4.25$$

Or, using the column method we used in example 2:

$$\begin{array}{rl} & x + y = 7 \\ + & x - y = 1.5 \\ \hline & 2x + 0y = 8.5 \end{array}$$

Again we get $2x = 8.5$, or $x = 4.25$. To find a corresponding value for y, we plug our value for x into either equation and isolate our unknown. In this example, we'll plug it into the first equation:

$$\begin{aligned} 4.25 + y &= 7 \qquad\qquad subtract\ 4.25\ from\ both\ sides: \\ y &= 2.75 \end{aligned}$$

Andrew paddles at 4.25 miles per hour. The river moves at 2.75 miles per hour.

Solving Linear Systems Using Subtraction of Equations

Another, very similar method for solving systems is subtraction. When the $x-$ or $y-$coefficients in both equations are the same (including the sign) instead of being opposites, you can **subtract** one equation from the other.

If you look again at Example 3, you can see that the coefficient for x in both equations is +1. Instead of adding the two equations together to get rid of the ys, you could have subtracted to get rid of the xs:

$$(x + y) - (x - y) = 7 - 1.5 \Rightarrow 2y = 5.5 \Rightarrow y = 2.75$$

or...

$$\begin{array}{rl} & x + y = 7 \\ - & (x - y) = -1.5 \\ \hline & 0x + 2y = 5.5 \end{array}$$

So again we get $y = 2.75$, and we can plug that back in to determine x.

The method of subtraction is just as straightforward as addition, so long as you remember the following:

- Always put the equation you are subtracting in parentheses, and distribute the negative.
- Don't forget to **subtract** the numbers on the right-hand side.

- Always remember that subtracting a negative is the same as adding a positive.

Example 4

Peter examines the coins in the fountain at the mall. He counts 107 coins, all of which are either pennies or nickels. The total value of the coins is $3.47. How many of each coin did he see?

Solution

We have 2 types of coins, so let's call the number of pennies x and the number of nickels y. The total value of all the pennies is just x, since they are worth $1¢$ each. The total value of the nickels is $5y$. We are given two key pieces of information to make our equations: the number of coins and their value in cents.

$$\begin{array}{lll} \#\text{ of coins equation}: & x+y=107 & (number\ of\ pennies)+(number\ of\ nickels) \\ \text{value equation}: & x+5y=347 & pennies\ are\ worth\ 1¢,\ nickels\ are\ worth\ 5¢. \end{array}$$

We'll jump straight to subtracting the two equations:

$$\begin{aligned} x+y&=107 \\ -\quad(x+5y)&=-347 \\ \hline -4y&=-240 \\ y&=60 \end{aligned}$$

Substituting this value back into the first equation:

$$\begin{aligned} x+60&=107 && subtract\ 60\ from\ both\ sides: \\ x&=47 \end{aligned}$$

So Peter saw 47 pennies (worth 47 cents) and 60 nickels (worth $3.00) making a total of $3.47.

Solving Linear Systems Using Multiplication

So far, we've seen that the elimination method works well when the coefficient of one variable happens to be the same (or opposite) in the two equations. But what if the two equations don't have any coefficients the same?

It turns out that we can still use the elimination method; we just have to *make* one of the coefficients match. We can accomplish this by multiplying one or both of the equations by a constant.

Here's a quick review of how to do that. Consider the following questions:

4. If 10 apples cost \$5, how much would 30 apples cost?

5. If 3 bananas plus 2 carrots cost \$4, how mush would 6 bananas plus 4 carrots cost?

If you look at the first equation, it should be obvious that each apple costs \$0.50. So 30 apples should cost \$15.00.

The second equation is trickier; it isn't obvious what the individual price for either bananas or carrots is. Yet we know that the answer to question 2 is \$8.00. How?

If we look again at question 1, we see that we can write an equation: $10a = 5$ (a being the cost of 1 apple). So to find the cost of 30 apples, we *could* solve for a and then multiply by 30—but we could also just multiply both sides of the equation by 3. We would get $30a = 15$, and that tells us that 30 apples cost \$15.

And we can do the same thing with the second question. The equation for this situation is $3b + 2c = 4$, and we can see that we need to solve for $(6b + 4c)$, which is simply 2 times $(3b + 2c)$! So algebraically, we are simply multiplying the entire equation by 2:

$$\begin{aligned} 2(3b + 2c) &= 2 \cdot 4 \qquad\qquad \textit{distribute and multiply}: \\ 6b + 4c &= 8 \end{aligned}$$

So when we multiply an equation, all we are doing is multiplying every term in the equation by a fixed amount.

Solving a Linear System by Multiplying One Equation

If we can multiply every term in an equation by a fixed number (a **scalar**), that means we can use the addition method on a whole new set of linear systems. We can manipulate the equations in a system to ensure that the coefficients of one of the variables match.

This is easiest to do when the coefficient as a variable in one equation is a multiple of the coefficient in the other equation.

Example 5

Solve the system:

$$7x + 4y = 17$$
$$5x - 2y = 11$$

Solution

You can easily see that if we multiply the second equation by 2, the coefficients of y will be +4 and -4, allowing us to solve the system by addition:

2 times equation 2:

$$\begin{array}{rl} 10x - 4y = 22 & \qquad \textit{now add to equation one}: \\ + \;\; (7x + 4y) = 17 & \\ \hline 17x = 34 & \end{array}$$

$$\textit{divide by 17 to get}: \qquad x = 2$$

Now simply substitute this value for x back into equation 1:

$$\begin{aligned} 7 \cdot 2 + 4y &= 17 && since\ 7 \times 2 = 14,\ subtract\ 14\ from\ both\ sides: \\ 4y &= 3 && divide\ by\ 4: \\ y &= 0.75 \end{aligned}$$

Example 6

Anne is rowing her boat along a river. Rowing downstream, it takes her 2 minutes to cover 400 yards. Rowing upstream, it takes her 8 minutes to travel the same 400 yards. If she was rowing equally hard in both directions, calculate, in yards per minute, the speed of the river and the speed Anne would travel in calm water.

Solution

Step one: first we convert our problem into equations. We know that *distance traveled* is equal to *speed* $\times$ *time*. We have two unknowns, so we'll call the speed of the river x, and the speed that Anne rows at y. When traveling downstream, her total speed is her rowing speed plus the speed of the river, or $(x + y)$. Going upstream, her speed is hindered by the speed of the river, so her speed upstream is $(x - y)$.

Downstream $2(x + y) = 400$ Equation:

Upstream $8(x - y) = 400$ Equation:

Distributing gives us the following system:

$$\begin{aligned} 2x + 2y &= 400 \\ 8x - 8y &= 400 \end{aligned}$$

Right now, we can't use the method of elimination because none of the coefficients match. But if we multiplied the top equation by 4, the coefficients of y would be +8 and -8. Let's do that:

$$
\begin{array}{r}
8x + 8y = 1,600 \\
+ \quad (8x - 8y) = 400 \\
\hline
16x = 2,000
\end{array}
$$

Now we divide by 16 to obtain $x = 125$.

Substitute this value back into the first equation:

$$
\begin{aligned}
2(125 + y) &= 400 && \textit{divide both sides by } 2: \\
125 + y &= 200 && \textit{subtract } 125 \textit{ from both sides}: \\
y &= 75
\end{aligned}
$$

Anne rows at 125 yards per minute, and the river flows at 75 yards per minute.

Solving a Linear System by Multiplying Both Equations

So what do we do if none of the coefficients match and none of them are simple multiples of each other? We do the same thing we do when we're adding fractions whose denominators aren't simple multiples of each other. Remember that when we add fractions, we have to find a **lowest common denominator**—that is, the lowest common multiple of the two denominators—and sometimes we have to rewrite not just one, but both fractions to get them to have a common denominator. Similarly, sometimes we have to multiply both equations by different constants in order to get one of the coefficients to match.

Example 7

Andrew and Anne both use the I-Haul truck rental company to move their belongings from home to the dorm rooms on the University of Chicago campus. I-Haul has a charge per day and an additional charge per mile. Andrew travels from San Diego, California, a distance of 2060 miles in five days. Anne travels 880 miles from Norfolk, Virginia, and it takes her three days. If Anne pays $840 and Andrew pays $1845, what does I-Haul charge

a) *per day?*

b) *per mile traveled?*

Solution

First, we'll set up our equations. Again we have 2 unknowns: the **daily rate** (we'll call this x), and the **per-mile rate** (we'll call this y).

Anne's $3x + 880y = 840$ equation:

Andrew's $5x + 2060y = 1845$ Equation:

We can't just multiply a single equation by an integer number in order to arrive at matching coefficients. But if we look at the coefficients of x (as they are easier to deal with than the coefficients of y), we see that they both have a common multiple of 15 (in fact 15 is the ***lowest common multiple***). So we can multiply both equations.

Multiply the top equation by 5:

$$15x + 4400y = 4200$$

Multiply the lower equation by 3:

$$15x + 6180y = 5535$$

Subtract:

$$\begin{array}{rr} & 15x + 4400y = 4200 \\ - & (15x + 6180y) = 5535 \\ \hline & -1780y = -1335 \end{array}$$

Divide by $-1780:\ \ y = 0.75$

Substitute this back into the top equation:

$$\begin{aligned} 3x + 880(0.75) &= 840 \quad \textit{since } 880 \times 0.75 = 660, \ \textit{subtract } 660 \ \textit{from both sides}: \\ 3x &= 180 \quad \textit{divide both sides by } 3 \\ x &= 60 \end{aligned}$$

I-Haul charges $60 per day plus $0.75 per mile.

Comparing Methods for Solving Linear Systems

Now that we've covered the major methods for solving linear equations, let's review them. For simplicity, we'll look at them in table form. This should help you decide which method would be best for a given situation.

Method:	**Best used when you...**	**Advantages:**	**Comment:**
Graphing	...don't need an accurate answer.	Often easier to see number and quality of intersections on a graph. With a graphing calculator, it can be the fastest method since you don't have to do any computation.	Can lead to imprecise answers with non-integer solutions.
Substitution	...have an *explicit* equation for one variable (e.g. $y = 14x + 2$)	Works on all systems. Reduces the system to one variable, making it easier to solve.	You are not often given explicit functions in systems problems, so you may have to do extra work to get one of the equations into that form.
Elimination by Addition or Subtraction	...have matching coefficients for one variable in both equations.	Easy to combine equations to eliminate one variable. Quick to solve.	It is not very likely that a given system will have matching coefficients.
Elimination by Multiplication and then Addition and Subtraction	...do not have any variables defined explicitly or any matching	Works on all systems. Makes it possible to combine equations to eliminate one variable.	Often more algebraic manipulation is needed to prepare the equations.

	coefficients.		

The table above is only a guide. You might prefer to use the graphical method for every system in order to better understand what is happening, or you might prefer to use the multiplication method even when a substitution would work just as well.

Example 8

Two angles are ***complementary*** *when the sum of their angles is* 90°. *Angles* A *and* B *are complementary angles, and twice the measure of angle* A *is* 9° *more than three times the measure of angle* B. *Find the measure of each angle.*

Solution

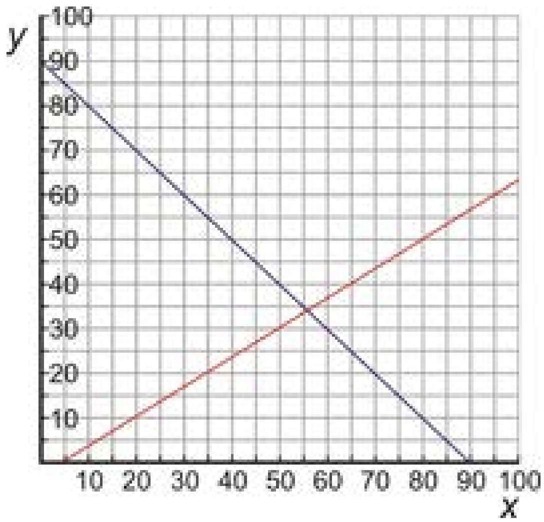
First we write out our 2 equations. We will use x to be the measure of angle A and y to be the measure of angle B. We get the following system:

$$x + y = 90$$
$$2x = 3y + 9$$

First, we'll solve this system with the graphical method. For this, we need to convert the two equations to $y = mx + b$ form:

$$x + y = 90 \qquad \Rightarrow y = -x + 90$$
$$2x = 3y + 9 \qquad \Rightarrow y = \frac{2}{3}x - 3$$

The first line has a slope of -1 and a $y-$intercept of 90, and the second line has a slope of $\frac{2}{3}$ and a $y-$intercept of -3. The graph looks like this:



In the graph, it appears that the lines cross at around $x = 55, y = 35$, but it is difficult to tell exactly! Graphing by hand is not the best method in this case!

Next, we'll try solving by substitution. Let's look again at the system:

$$\begin{aligned} x + y &= 90 \\ 2x &= 3y + 9 \end{aligned}$$

We've already seen that we can start by solving either equation for y, so let's start with the first one:

$$y = 90 - x$$

Substitute into the second equation:

$$\begin{aligned} 2x &= 3(90 - x) + 9 && \textit{distribute the } 3: \\ 2x &= 270 - 3x + 9 && \textit{add } 3x \textit{ to both sides}: \\ 5x &= 270 + 9 = 279 && \textit{divide by } 5: \\ x &= 55.8^\circ \end{aligned}$$

Substitute back into our expression for y:

$y = 90 - 55.8 = 34.2^\circ$

Angle A **measures** 55.8°; **angle** B **measures** 34.2°.

Finally, we'll try solving by elimination (with multiplication):

Rearrange equation one to standard form:

$x + y = 90 \qquad \Rightarrow 2x + 2y = 180$

Multiply equation two by 2:

$2x = 3y + 9 \qquad \Rightarrow 2x - 3y = 9$

Subtract:

$$\begin{array}{rr} & 2x + 2y = 180 \\ - & (2x - 3y) = -9 \\ \hline & 5y = 171 \end{array}$$

Divide by 5 to obtain $y = 34.2^\circ$

Substitute this value into the very first equation:

$$\begin{aligned} x + 34.2 &= 90 \qquad \qquad subtract\ 34.2\ from\ both\ sides: \\ x &= 55.8^\circ \end{aligned}$$

Angle A **measures** 55.8°; **angle** B **measures** 34.2°.

Even though this system looked ideal for substitution, the method of multiplication worked well too. Once the equations were rearranged properly, the solution was quick to find. You'll need to decide yourself which method to use in each case you see from now on. Try to master all the techniques, and recognize which one will be most efficient for each system you are asked to solve.

The following Khan Academy video contains three examples of solving systems of equations using addition and subtraction as well as multiplication (which is the next topic): http://www.youtube.com/watch?v=nok99JOhcjo (9:57). (Note that the narrator is not always careful about showing his work, and you should try to be neater in your mathematical writing.)

For even more practice, we have this video. One common type of problem involving systems of equations (especially on standardized tests) is "age problems." In the following video the narrator shows two examples of age problems, one involving a single person and one involving two people. Khan Academy Age Problems (7:13)

2a. Linear Inequalities in Two Variables

Objectives

- Graph linear inequalities in one variable on the coordinate plane.
- Graph linear inequalities in two variables.
- Solve real-world problems using linear inequalities.

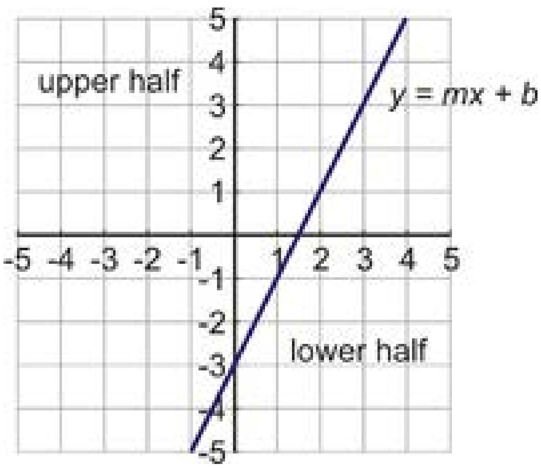
Concept

Introduction

Yasmeen is selling handmade bracelets for $5 each and necklaces for $7 each. How many of both does she need to sell to make at least $100?

A **linear inequality** in two variables takes the form $y > mx + b$ or $y < mx + b$. Linear inequalities are closely related to graphs of straight lines; recall that a straight line has the equation $y = mx + b$.

When we graph a line in the coordinate plane, we can see that it divides the plane in half:



The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

$>$ The solution set is the half plane above the line.

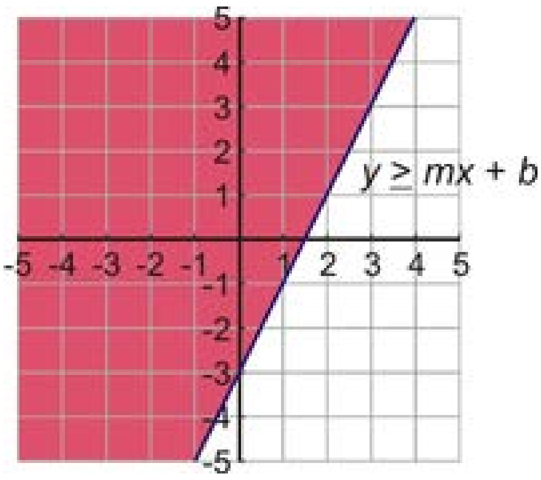
$\geq$ The solution set is the half plane above the line and also all the points on the line.

$<$ The solution set is the half plane below the line.

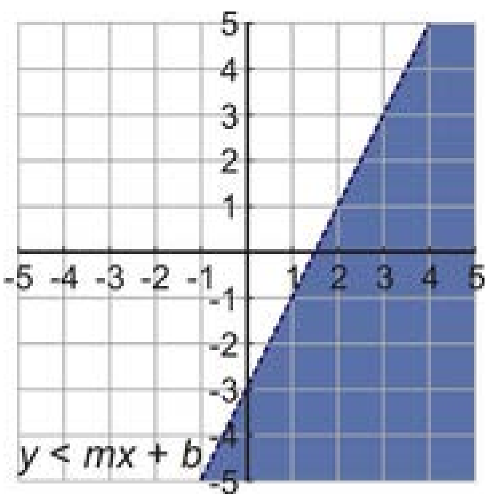
$\leq$ The solution set is the half plane below the line and also all the points on the line.

For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.

Here are some examples of linear inequality graphs. This is a graph of $y \geq mx + b$; the solution set is the line and the half plane above the line.



This is a graph of $y < mx + b$; the solution set is the half plane above the line, not including the line itself.



Graph Linear Inequalities in One Variable in the Coordinate Plane

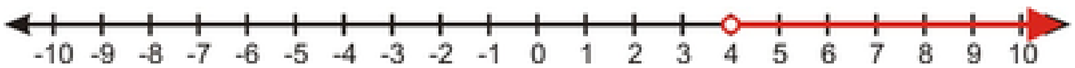
In the last few sections we graphed inequalities in one variable on the number line. We can also graph inequalities in one variable on the coordinate plane. We just need to remember that when we graph an equation of the type $x = a$ we get a vertical line, and when we graph an equation of the type $y = b$ we get a horizontal line.

Example 1

Graph the inequality $x > 4$ *on the coordinate plane.*

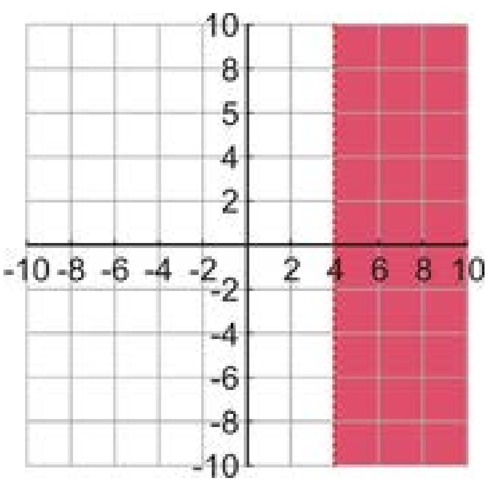
Solution

First let's remember what the solution to $x > 4$ looks like on the number line.



The solution to this inequality is the set of all real numbers x that are bigger than 4, not including 4. The solution is represented by a line.

In two dimensions, the solution still consists of all the points to the right of $x = 4$, but for all possible $y-$values as well. This solution is represented by the half plane to the right of $x = 4$. (You can think of it as being like the solution graphed on the number line, only stretched out vertically.)



The line $x = 4$ is dashed because the equals sign is not included in the inequality, meaning that points on the line are not included in the solution.

Example 2

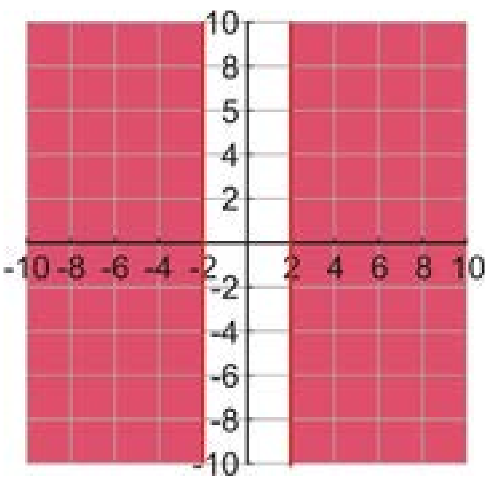
Graph the inequality . $|x| \geq 2$

Solution

The absolute value $|x| \geq 2$ inequality can be re-written as a compound inequality:

$x \leq -2 \quad \text{or} \quad x \geq 2$

In other words, the solution is all the coordinate points for which the value of x is smaller than or equal to -2 **or** greater than or equal to 2. The solution is represented by the plane to the left of the vertical line $x = -2$ and the plane to the right of line $x = 2$.



Both vertical lines are solid because points on the lines are included in the solution.

Example 3

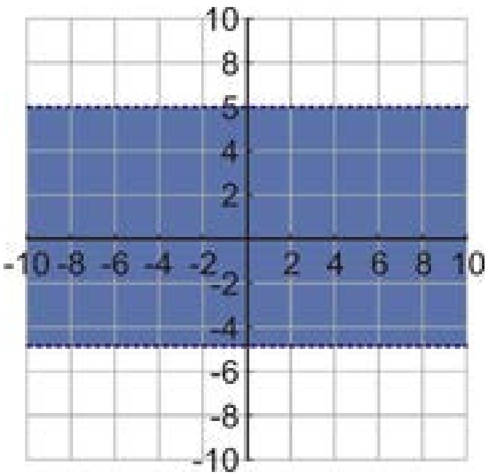
Graph the $|y| < 5$ inequality

Solution

The absolute value inequality $|y| < 5$ can be re-written as $-5 < y < 5$. This is a compound inequality which can be expressed as

$y > -5 \quad \text{and} \quad y < 5$

In other words, the solution is all the coordinate points for which the value of y is larger than -5 **and** smaller than 5. The solution is represented by the plane between the horizontal lines $y = -5$ and $y = 5$.



Both horizontal lines are dashed because points on the lines are not included in the solution.

Graph Linear Inequalities in Two Variables

The general procedure for graphing inequalities in two variables is as follows:

- Re-write the inequality in slope-intercept form: $y = mx + b$. Writing the inequality in this form lets you know the direction of the inequality.
- Graph the line of the equation $y = mx + b$ using your favorite method (plotting two points, using slope and $y-$intercept, using $y-$intercept and another point, or whatever is easiest). Draw the line as a dashed line if the equals sign is not included and a solid line if the equals sign is included.
- Shade the half plane above the line if the inequality is “greater than.” Shade the half plane under the line if the inequality is “less than.”

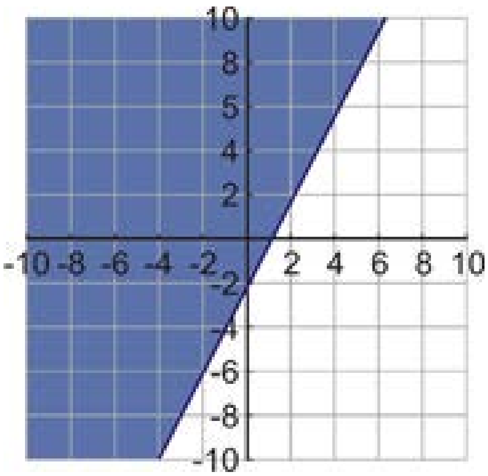
Example 4

$$y \geq 2x - 3$$

Graph the inequality .

Solution

The inequality is already written in slope-intercept form, so it's easy to graph. First we graph the line $y = 2x - 3$; then we shade the half-plane above the line. The line is solid because the inequality includes the equals sign.



Example 5

Graph the inequality $5x - 2y > 4$.

Solution

First we need to rewrite the inequality in slope-intercept form:

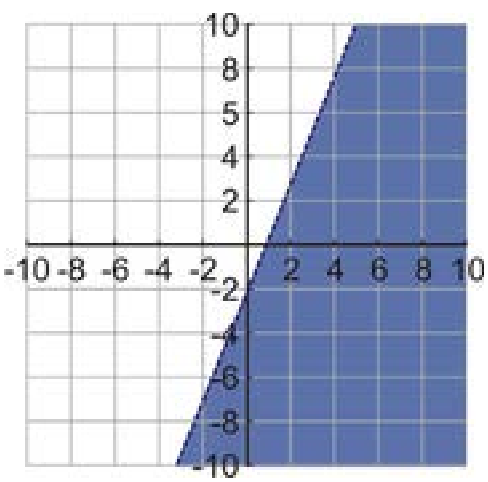
$$\begin{aligned} -2y &> -5x + 4 \\ y &< \frac{5}{2}x - 2 \end{aligned}$$

Notice that the inequality sign changed direction because we divided by a negative number.

To graph the equation, we can make a table of values:

x	y
-2	$\frac{5}{2}(-2) - 2 = -7$
0	$\frac{5}{2}(0) - 2 = -2$
2	$\frac{5}{2}(2) - 2 = 3$

After graphing the line, we shade the plane **below** the line because the inequality in slope-intercept form is **less than**. The line is dashed because the inequality does not include an equals sign.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Example 8

A retailer sells two types of coffee beans. One type costs \$9 per pound and the other type costs \$7 per pound. Find all the possible amounts of the two different coffee beans that can be mixed together to get a quantity of coffee beans costing \$8.50 or less.

Solution

Let $x =$ weight of \$9 per pound coffee beans in pounds.

Let $y =$ weight of \$7 per pound coffee beans in pounds.

The cost of a pound of coffee blend is given $9x + 7y$ by .

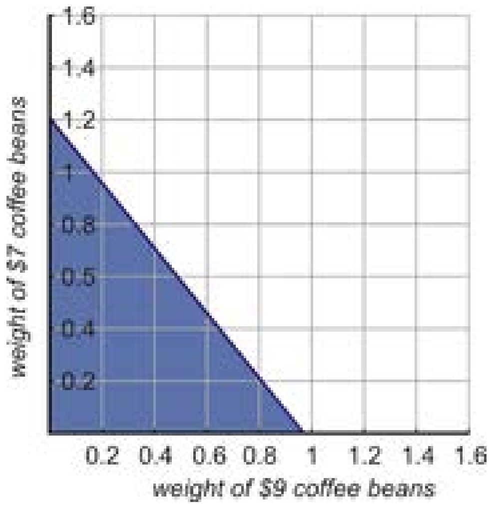
We are looking for the mixtures that cost \$8.50 or less. We write the $9x + 7y \leq 8.50$ inequality .

Since this inequality is in standard form, it's easiest to graph it by finding the $x-$ $y-$ and intercepts. When $x = 0$, we have $7y = 8.50$ or $y = \frac{8.50}{7} \approx 1.21$. When $y = 0$, we have $9x = 8.50$ or $x = \frac{8.50}{9} \approx 0.94$. We can then graph the line that includes those two points.

Now we have to figure out which side of the line to shade. In $y-$intercept form, we shade the area **below** the line when the inequality is "less than." But in standard form that's not always true. We could convert the inequality to $y-$intercept form to find out which side to shade, but there is another way that can be easier.

The other method, which works for any linear inequality in any form, is to plug a random point into the inequality and see if it makes the inequality true. Any point that's not on the line will do; the point (0, 0) is usually the most convenient.

In this case, plugging in 0 x for and y would give us $9(0) + 7(0) \leq 8.50$, which is true. That means we should shade the half of the plane that includes (0, 0). If plugging in (0, 0) gave us a false inequality, that would mean that the solution set is the part of the plane that does *not* contain (0, 0).



Notice also that in this graph we show only the first quadrant of the coordinate plane. That's because weight values in the real world are always nonnegative, so points outside the first quadrant don't represent real-world solutions to this problem.

Example 9

Julius has a job as an appliance salesman. He earns a commission of $60 for each washing machine he sells and $130 for each refrigerator he sells. How many washing machines and refrigerators must Julius sell in order to make $1000 or more in commissions?

Solution

Let $x =$ number of washing machines Julius sells.

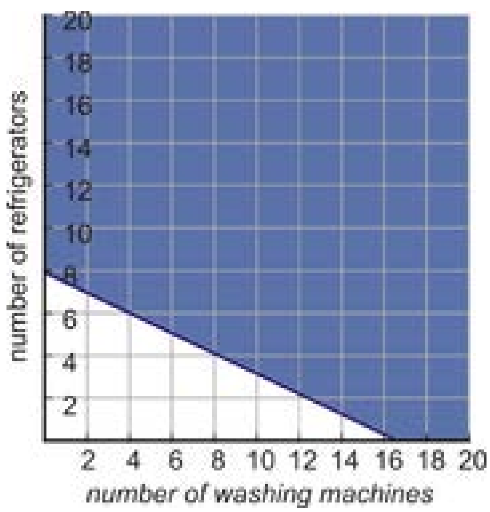
Let $y =$ number of refrigerators Julius sells.

The total commission $60x + 130y$ is .

We're looking for a total commission of $1000 or more, so we write the inequality $60x + 130y \geq 1000$.

Once again, we can do this most easily by finding the $x-$ $y-$ and intercepts. When $x = 0$, we have $130y = 1000$, or $y = \frac{1000}{30} \approx 7.69$. When $y = 0$, we have $60x = 1000$, or $x = \frac{1000}{60} \approx 16.67$.

We draw a solid line connecting those points, and shade above the line because the inequality is “greater than.” We can check this by plugging in the point (0, 0): selling 0 washing machines and 0 refrigerators would give Julius a commission of $0, which is *not* greater than or equal to $1000, so the point (0, 0) is *not* part of the solution; instead, we want to shade the side of the line that does *not* include it.



Notice also that we show only the first quadrant of the coordinate plane, because Julius’s commission should be nonnegative.

The video at http://www.youtube.com/watch?v=7629PsZLP1A&feature=related contains more examples of real-world problems using inequalities in two variables.

2b. Systems of Linear Inequalities

Objectives

- Graph linear inequalities in two variables.
- Solve systems of linear inequalities.
- Solve optimization problems.

Concept

Introduction

In the last chapter you learned how to graph a linear inequality in two variables. To do that, you graphed the equation of the straight line on the coordinate plane. The line was solid for $\leq$ or $\geq$ signs (where the equals sign is included), and the line was dashed for < or > signs (where the equals sign is not included). Then you shaded above the line (if the inequality began with $y >$ or $y \geq$) or below the line (if it began with $y <$ or $y \leq$).

In this section, we'll see how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph, and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes.

Let's start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example 1

Solve the following system:

$$\begin{aligned} 2x + 3y &\leq 18 \\ x - 4y &\leq 12 \end{aligned}$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection *regions* of the solution.

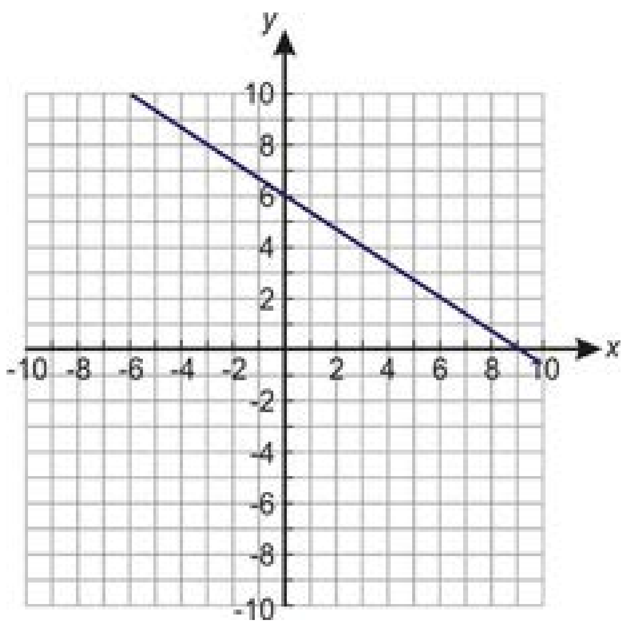
First, let's rewrite each equation in slope-intercept form. (Remember that this form makes it easier to tell which region of the coordinate plane to shade.) Our system becomes

$$\begin{aligned} 3y &\leq -2x + 18 \\ -4y &\leq -x + 12 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &\leq -\frac{2}{3}x + 6 \\ y &\geq \frac{x}{4} - 3 \end{aligned}$$

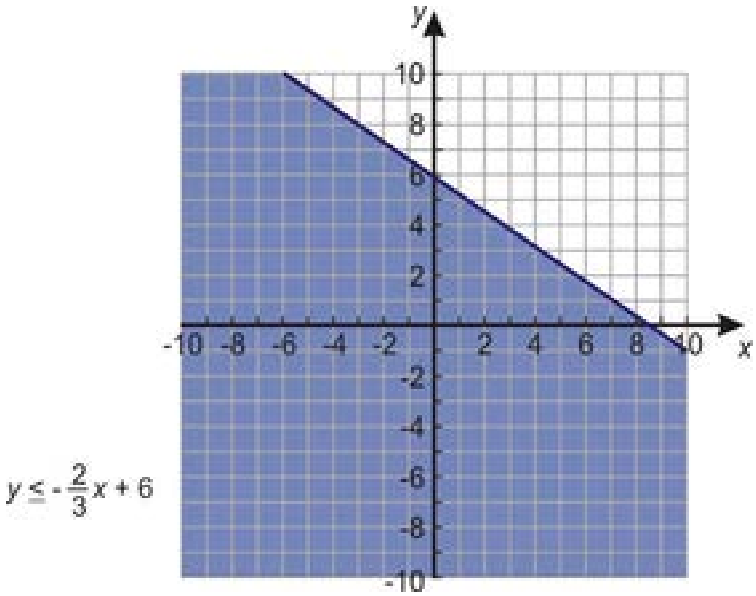
Notice that the inequality sign in the second equation changed because we divided by a negative number!

For this first example, we'll graph each inequality separately and then combine the results.

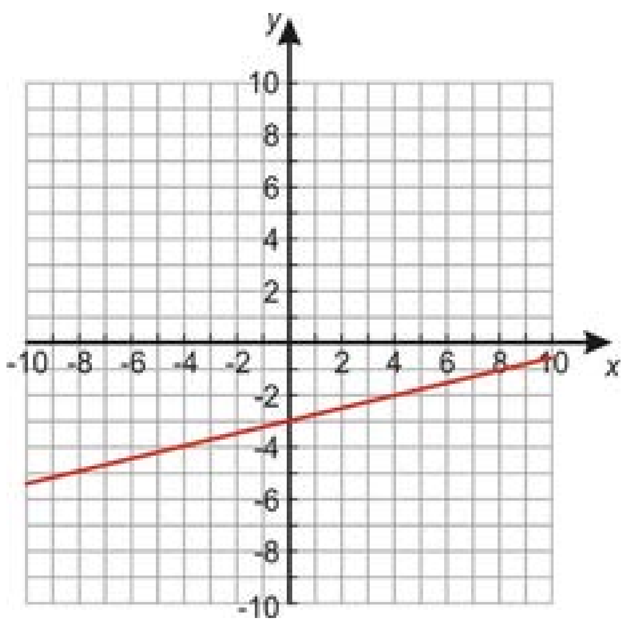
Here's the graph of the first inequality:



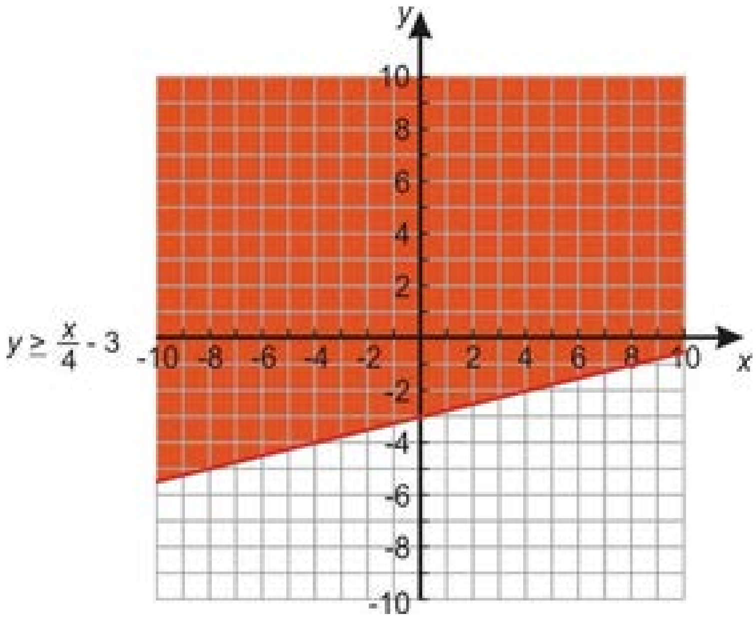
The line is solid because the equals sign is included in the inequality. Since the inequality is **less** than or equal to, we shade **below** the line.



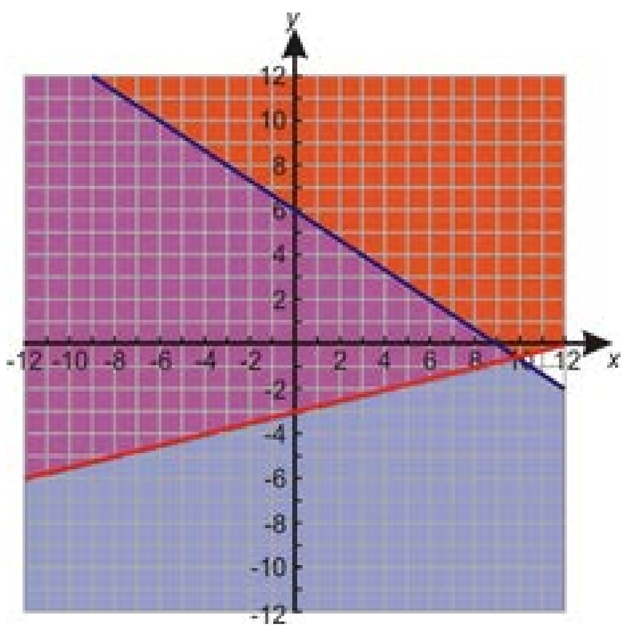
And here’s the graph of the second inequality:



The line is solid again because the equals sign is included in the inequality. We now shade **above** the line because y is **greater** than or equal to.



When we combine the graphs, we see that the blue and red shaded regions overlap. The area where they overlap is the area where both inequalities are true. Thus that area (shown below in purple) is the solution of the system.



The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

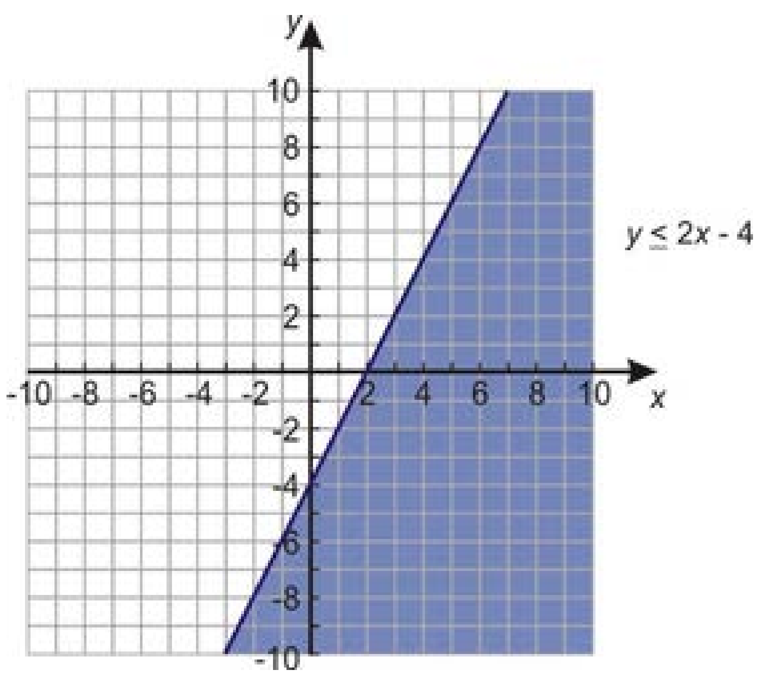
Example 2

There are also situations where a system of inequalities has no solution. For example, let's solve this system.

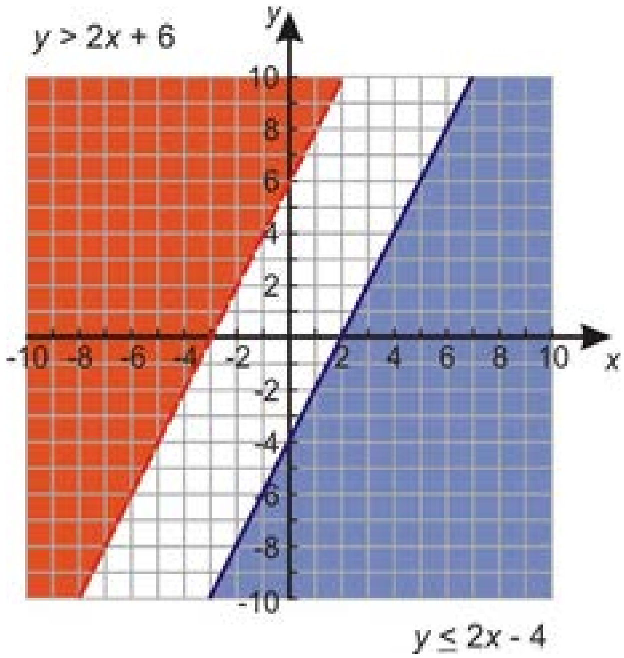
$y \leq 2x - 4$
$y > 2x + 6$

Solution

We start by graphing the first line. The line will be solid because the equals sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equals sign is not included in the inequality. We must shade upward because y is greater than.



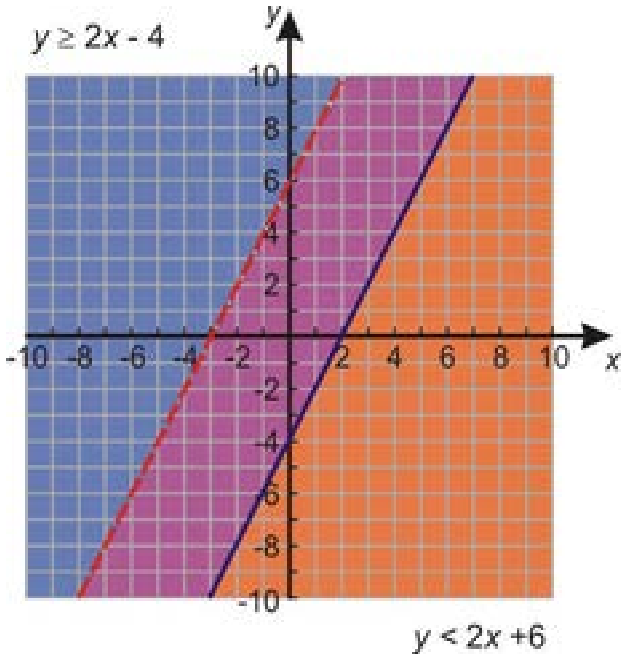
It doesn’t look like the two shaded regions overlap at all. The two lines have the same slope, so we know they are parallel; that means that the regions indeed won’t ever overlap since the lines won’t ever cross. So this system of inequalities has no solution.

But a system of inequalities can sometimes have a solution even if the lines are parallel. For example, what happens if we swap the directions of the inequality signs in the system we just graphed?

To graph the system

$$\begin{aligned} y &\geq 2x - 4 \\ y &< 2x + 6 \end{aligned},$$

we draw the same lines we drew for the previous system, but we shade *upward* for the first inequality and *downward* for the second inequality. Here is the result:



You can see that this time the shaded regions overlap. The area between the two lines is the solution to the system.

Graph a System of More Than Two Linear Inequalities

When we solve a system of just two linear inequalities, the solution is always an **unbounded** region—one that continues infinitely in at least one direction. But if we put together a system of more than two inequalities, sometimes we can get a solution that is **bounded**—a finite region with three or more sides.

Let's look at a simple example.

Example 3

Find the solution to the following system of inequalities.

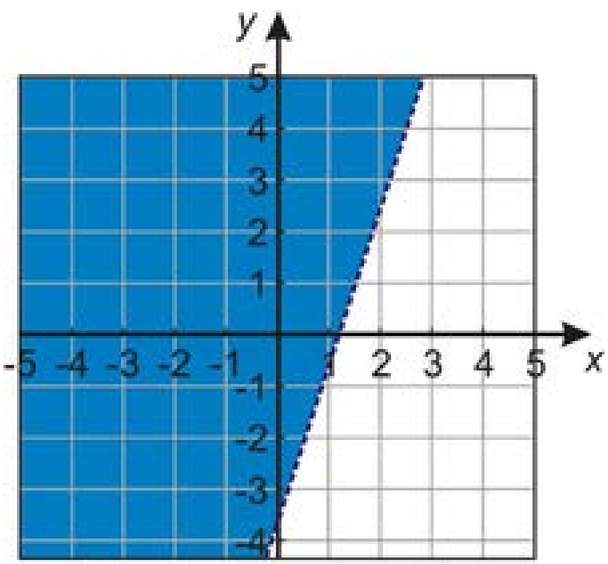
$$\begin{aligned} 3x - y &< 4 \\ 4y + 9x &< 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solution

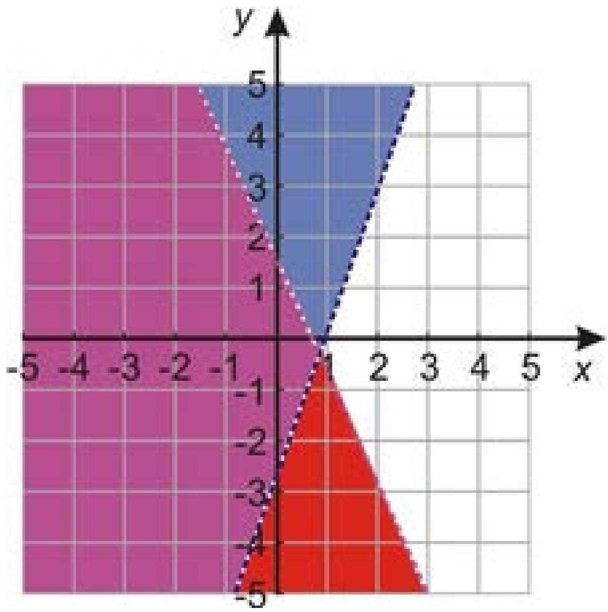
Let's start by writing our inequalities in slope-intercept form.

$$y > 3x - 4$$
$$y < -\frac{9}{4}x + 2$$
$$x \geq 0$$
$$y \geq 0$$

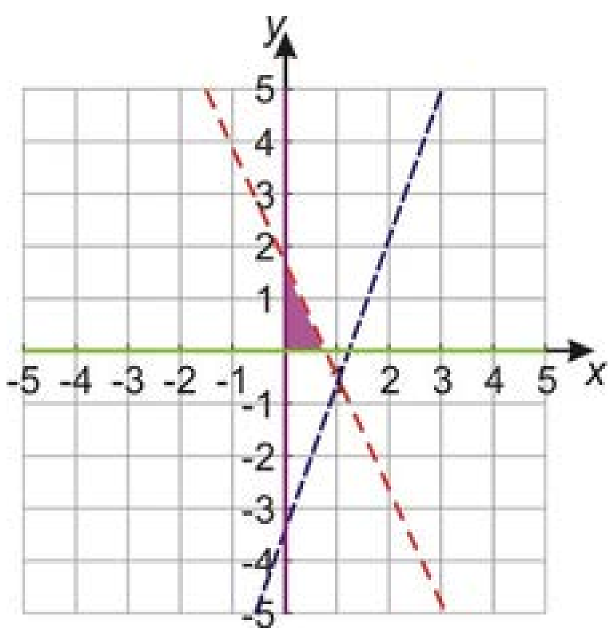
Now we can graph each line and shade appropriately. First we $y > 3x - 4$ graph :



Next we $y < -\frac{9}{4}x + 2$ graph :



Finally we graph $x \geq 0$ and $y \geq 0$, and we're left with the region below; this is where all four inequalities overlap.



The solution is **bounded** because there are lines on all sides of the solution region. In other words, the solution region is a bounded geometric figure, in this case a triangle.

Notice, too, that only three of the lines we graphed actually form the boundaries of the region. Sometimes when we graph multiple inequalities, it turns out that some of them don't affect the overall solution; in this case, the solution would be the same even if we'd left out the inequality $y > 3x - 4$. That's because the solution region of the system formed by the other three inequalities is completely contained within the solution region of that fourth inequality; in other words, any solution to the other three inequalities is *automatically* a solution to that one too, so adding that inequality doesn't narrow down the solution set at all.

But that wasn't obvious until we actually drew the graph!

Solve Real-World Problems Using Systems of Linear Inequalities

A lot of interesting real-world problems can be solved with systems of linear inequalities.

For example, you go to your favorite restaurant and you want to be served by your best friend who happens to work there. However, your friend only waits tables in a certain region of the restaurant. The restaurant is also known for its great views, so you want to sit in a certain area of the restaurant that offers a good view. Solving a system of linear inequalities will allow you to find the area in the restaurant where you can sit to get the best view and be served by your friend.

Often, systems of linear inequalities deal with problems where you are trying to find the best possible situation given a set of constraints. Most of these application problems fall in a category called **linear programming** problems.

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the *best* possible value under those conditions. A typical example would be taking the limitations of materials and labor at a factory, then determining the best production levels for maximal profits under those conditions. These kinds of problems are used every day in the organization and allocation of resources. These real-life systems can have dozens or hundreds of variables, or more. In this section, we'll only work with the simple two-variable linear case.

The general process is to:

- Graph the inequalities (called **constraints**) to form a bounded area on the coordinate plane (called **the feasibility region**).
- Figure out the coordinates of the corners (or vertices) of this feasibility region by solving the system of equations that applies to each of the intersection points.
- Test these corner points in the formula (called the **optimization equation**) for which you're trying to find the **maximum** or **minimum** value.

Example 4

If $z = 2x + 5y$*, find the maximum and minimum values of* z *given these constraints:*

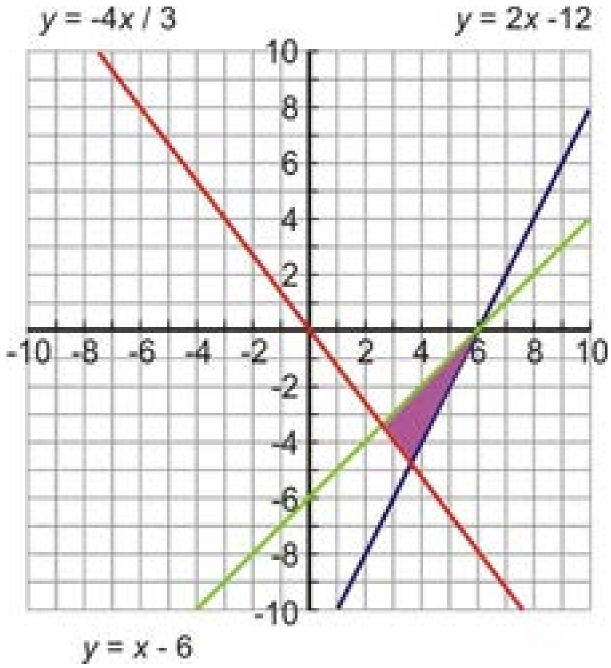
$$\begin{aligned} 2x - y &\leq 12 \\ 4x + 3y &\geq 0 \\ x - y &\leq 6 \end{aligned}$$

Solution

First, we need to find the solution to this system of linear inequalities by graphing and shading appropriately. To graph the inequalities, we rewrite them in slope-intercept form:

$$\begin{aligned} &y \geq 2x - 12 \\ &y \geq -\frac{4}{3}x \\ &y \geq x - 6 \end{aligned}$$

These three linear inequalities are called the **constraints**, and here is their graph:



The shaded region in the graph is called the **feasibility region**. All possible solutions to the system occur in that region; now we must try to find the maximum and minimum values of the variable z within that region. In other words, which values of x and y within the feasibility region will give us the greatest and smallest overall values for the expression $2x + 5y$?

Fortunately, we don't have to test every point in the region to find that out. It just so happens that the minimum or maximum value of the optimization equation in a linear system like this will always be found at one of the vertices (the corners) of the feasibility region; we just have to figure out *which* vertices. So for each vertex—each point where two of the lines on the graph cross—we need to solve the system of just those two equations, and then find the value of z at that point.

The first system consists of the equations $y = 2x - 12$ and $y = -\frac{4}{3}x$. We can solve this system by substitution:

$$-\frac{4}{3}x = 2x - 12 \Rightarrow -4x = 6x - 36 \Rightarrow -10x = -36 \Rightarrow x = 3.6$$
$$y = 2x - 12 \Rightarrow y = 2(3.6) - 12 \Rightarrow y = -4.8$$

The lines intersect at the point (3.6, -4.8).

The second system consists of the equations $y = 2x - 12$ and $y = x - 6$. Solving this system by substitution:

$$x - 6 = 2x - 12 \Rightarrow 6 = x \Rightarrow x = 6$$
$$y = x - 6 \Rightarrow y = 6 - 6 \Rightarrow y = 6$$

The lines intersect at the point (6, 6).

The third system consists of the equations $y = -\frac{4}{3}x$ and $y = x - 6$. Solving this system by substitution:

$$x - 6 = -\frac{4}{3}x \Rightarrow 3x - 18 = -4x \Rightarrow 7x = 18 \Rightarrow x = 2.57$$
$$y = x - 6 \Rightarrow y = 2.57 - 6 \Rightarrow y = -3.43$$

The lines intersect at the point (2.57, -3.43).

So now we have three different points that might give us the maximum and minimum values for z. To find out which ones actually do give the maximum and minimum values, we can plug the points into the optimization equation $z = 2x + 5y$.

When we plug in (3.6, -4.8), we $z = 2(3.6) + 5(-4.8) = -16.8$ get .

When we plug in (6, 0), we $z = 2(6) + 5(0) = 12$ get .

When we plug in (2.57, -3.43), we $z = 2(2.57) + 5(-3.43) = -12.01$ get .

So we can see that **the point (6, 0) gives us the maximum possible value for z and the point (3.6, –4.8) gives us the minimum value.**

In the previous example, we learned how to apply the method of linear programming in the abstract. In the next example, we'll look at a real-life application.

Example 5

You have $10,000 to invest, and three different funds to choose from. The municipal bond fund has a 5% return, the local bank's CDs have a 7% return, and a high-risk account has an expected 10% return. To minimize risk, you decide not to invest any more than $1,000 in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. What's the best way to distribute your money given these constraints?

Solution

Let's define our variables:

x is the amount of money invested in the municipal bond at 5% return

y is the amount of money invested in the bank's CD at 7% return

$10000 - x - y$ is the amount of money invested in the high-risk account at 10% return

z is the total interest returned from all the investments, so $z = .05x + .07y + .1(10000 - x - y)$ or $z = 1000 - 0.05x - 0.03y$. This is the amount that we are trying to maximize. Our goal is to find the values of x and y that maximizes the value of z.

Now, let's write inequalities for the *constraints*:

You decide not to invest more than $1000 in the high-risk account—that means:

$10000 - x - y \leq 1000$

You need to invest at least three times as much in the municipal bonds as in the bank CDs—that means:

$3y \le x$

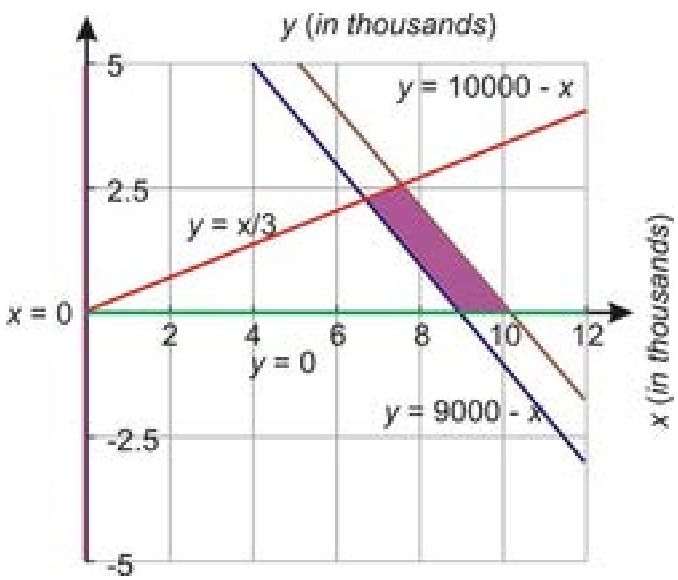
Also, you can't invest less than zero dollars in each account, so:

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \\ 10000 - x - y &\ge 0 \end{aligned}$$

To summarize, we must maximize the expression $z = 1000 - .05x - .03y$ using the constraints:

$$\begin{array}{lcl} 10000 - x - y \le 1000 & & y \ge 9000 - x \\ 3y \le x & & y \le \dfrac{x}{3} \\ x \ge 0 & \text{Or in slope-intercept form:} & x \ge 0 \\ y \ge 0 & & y \ge 0 \\ 10000 - x - y \ge 0 & & y \le 10000 - x \end{array}$$

Step 1: Find the solution region to the set of inequalities by graphing each line and shading appropriately. The following figure shows the overlapping region:



The purple region is the feasibility region where all the possible solutions can occur.

Step 2: Next we need to find the corner points of the feasibility region. Notice that there are four corners. To find their coordinates, we must pair up the relevant equations and solve each resulting system.

System 1:

$$y = \frac{x}{3}$$
$$y = 10000 - x$$

Substitute the first equation into the second equation:

$$\frac{x}{3} = 10000 - x \Rightarrow x = 30000 - 3x \Rightarrow 4x = 30000 \Rightarrow x = 7500$$
$$y = \frac{x}{3} \Rightarrow y = \frac{7500}{3} \Rightarrow y = 2500$$

The intersection point is (7500, 2500).

System 2:

$$y = \frac{x}{3}$$
$$y = 9000 - x$$

Substitute the first equation into the second equation:

$$\frac{x}{3} = 9000 - x \Rightarrow x = 27000 - 3x \Rightarrow 4x = 27000 \Rightarrow x = 6750$$
$$y = \frac{x}{3} \Rightarrow y = \frac{6750}{3} \Rightarrow y = 2250$$

The intersection point is (6750, 2250).

System 3:

$$\begin{aligned} y &= 0 \\ y &= 10000 - x \end{aligned}.$$

The intersection point is (10000, 0).

System 4:

$$\begin{aligned} y &= 0 \\ y &= 9000 - x \end{aligned}.$$

The intersection point is (9000, 0).

Step 3: In order to find the maximum value for z, we need to plug all the intersection points into the equation for z and find which one yields the largest number.

(7500, 2500): $z = 1000 - 0.05(7500) - 0.03(2500) = 550$

(6750, 2250): $z = 1000 - 0.05(6750) - 0.03(2250) = 595$

(10000, 0): $z = 1000 - 0.05(10000) - 0.03(0) = 500$

(9000, 0): $z = 1000 - 0.05(9000) - 0.03(0) = 550$

The maximum return on the investment of $595 occurs at the point (6750, 2250). This means that:

$6,750 is invested in the municipal bonds.

$2,250 is invested in the bank CDs.

$1,000 is invested in the high-risk account.

Graphing calculators can be very useful for problems that involve this many inequalities. The video at http://www.youtube.com/watch?v=__wAxkYmhvY shows a real-world linear programming problem worked through in detail on a graphing calculator, although the methods used there can also be used for pencil-and paper solving.

3a. Linear Equations in Standard Form

You've already encountered another useful form for writing linear equations: **standard form.** An equation in standard form is written $ax + by = c$, where a, b, and c are all integers and a is positive. (Note that the b in the standard form is different than the b in the slope-intercept form.)

One useful thing about standard form is that it allows us to write equations for vertical lines, which we can't do in slope-intercept form.

For example, let's look at the line that passes through points (2, 6) and (2, 9). How would we find an equation for that line in slope-intercept form?

First we'd need to find the slope: $m = \frac{9-6}{0-0} = \frac{3}{0}$. But that slope is undefined because we can't divide by zero. And if we can't find the slope, we can't use point-slope form either.

If we just graph the line, we can see that x equals 2 no matter what y is. There's no way to express that in slope-intercept or point-slope form, but in standard form we can just say that $x + 0y = 2$, or simply $x = 2$.

Converting to Standard Form

To convert an equation from another form to standard form, all you need to do is rewrite the equation so that all the variables are on one side of the equation and the coefficient of x is not negative.

Example 1

Rewrite the following equations in standard form:

a) $y = 5x - 7$

b) $y - 2 = -3(x + 3)$

c) $y = \frac{2}{3}x + \frac{1}{2}$

Solution

We need to rewrite each equation so that all the variables are on one side and the coefficient of x is not negative.

$y = 5x - 7$ a)

Subtract y from both sides to get . $0 = 5x - y - 7$

Add 7 to both sides to get $7 = 5x - y$.

Flip the equation around to put it in standard form: . $5x - y = 7$

$y - 2 = -3(x + 3)$ b)

Distribute the –3 on the right-hand-side to get . $y - 2 = -3x - 9$

Add $3x$ to both sides to get . $y + 3x - 2 = -9$

Add 2 to both sides to get $y + 3x = -7$. Flip that around to get . $3x + y = -7$

$y = \frac{2}{3}x + \frac{1}{2}$ c)

Find the common denominator for all terms in the equation – in this case that would be 6.

Multiply all terms in the equation by 6: $6\left(y = \frac{2}{3}x + \frac{1}{2}\right) \Rightarrow 6y = 4x + 3$

Subtract $6y$ from both sides: $0 = 4x - 6y + 3$

Subtract 3 from both sides: $-3 = 4x - 6y$

The equation in standard form is $4x - 6y = -3$.

3b. Graphing Equations in Standard Form

When an equation is in slope-intercept form or point-slope form, you can tell right away what the slope is. How do you find the slope when an equation is in standard form?

Well, you could rewrite the equation in slope-intercept form and read off the slope. But there's an even easier way. Let's look at what happens when we rewrite an equation in standard form.

Starting with the equation $ax + by = c$, we would subtract ax from both sides to get $by = -ax + c$. Then we would divide all terms by b and end up with $y = -\frac{a}{b}x + \frac{c}{b}$.

That means that the slope is $-\frac{a}{b}$ and the $y-$intercept is $\frac{c}{b}$. So next time we look at an equation in standard form, we don't have to rewrite it to find the slope; we know the slope is just $-\frac{a}{b}$, where a and b are the coefficients of x and y in the equation.

Example 2

Find the slope and the $y-$intercept of the following equations written in standard form.

$3x + 5y = 6$ a)

$2x - 3y = -8$ b)

$x - 5y = 10$ c)

Solution

$a = 3, b = 5$ a) , and $c = 6$, so the slope is $-\frac{a}{b} = -\frac{3}{5}$, and the $y-$intercept is $\frac{c}{b} = \frac{6}{5}$.

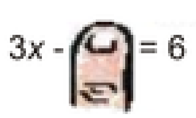
$a = 2, b = -3$ b) , and $c = -8$, so the slope is $-\frac{a}{b} = \frac{2}{3}$, and the $y-$intercept is $\frac{c}{b} = \frac{8}{3}$.

$a = 1, b = -5$ c) , and $c = 10$, so the slope is $-\frac{a}{b} = \frac{1}{5}$, and the $y-$intercept is $\frac{c}{b} = \frac{10}{-5} = -2$.

Once we've found the slope and $y-$intercept of an equation in standard form, we can graph it easily. But if we start with a graph, how do we find an equation of that line in standard form?

First, remember that we can also use the cover-up method to graph an equation in standard form, by finding the intercepts of the line. For example, let's graph the line given by the equation $3x - 2y = 6$.

To find the $x-$intercept, cover up the y term (remember, the $x-$intercept is where $y = 0$):



$3x = 6 \Rightarrow x = 2$

The $x-$ intercept is (2, 0).

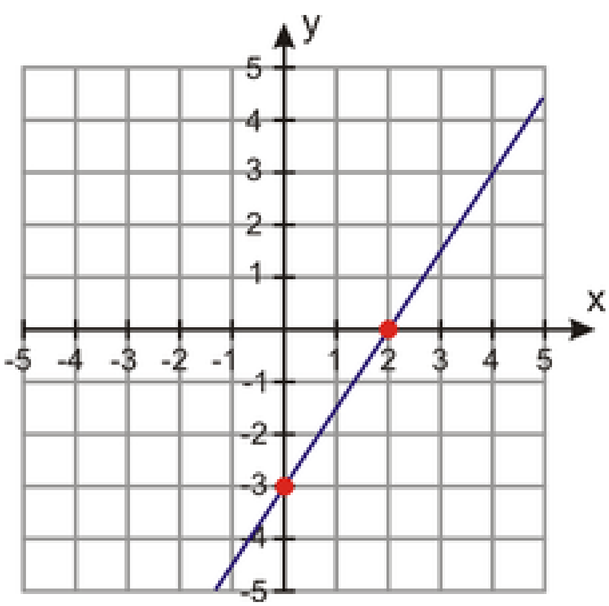
To find the $y-$intercept, cover up the x term (remember, the $y-$intercept is where $x = 0$):

 $-2y = 6$

$-2y = 6 \Rightarrow y = -3$

The $y-$intercept is (0, -3).

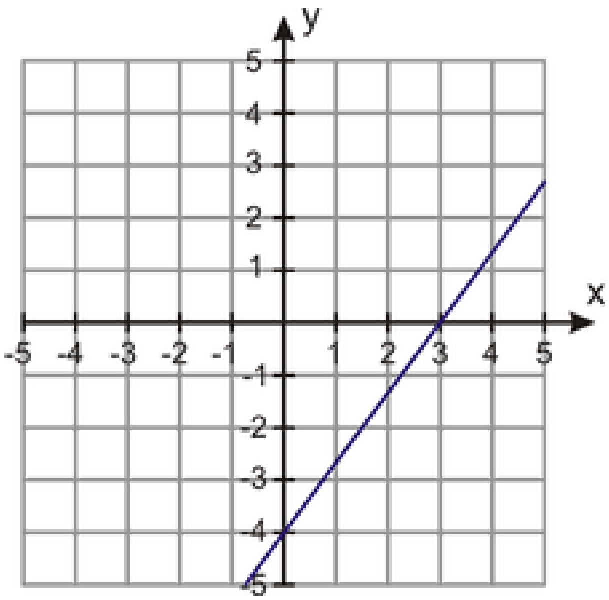
We plot the intercepts and draw a line through them that extends in both directions:

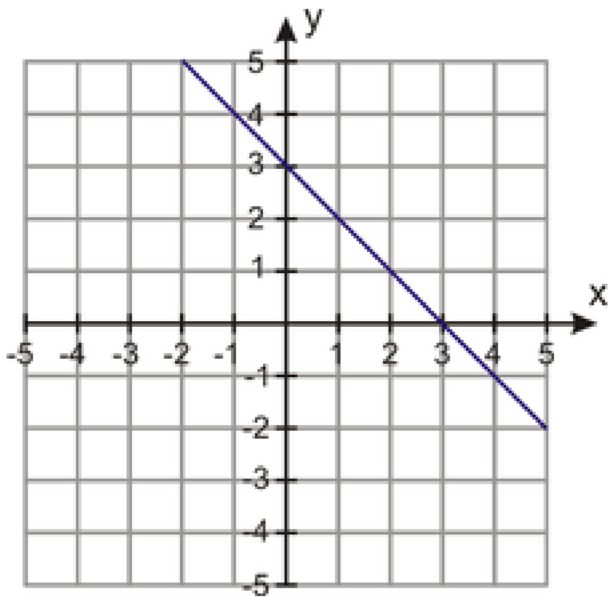


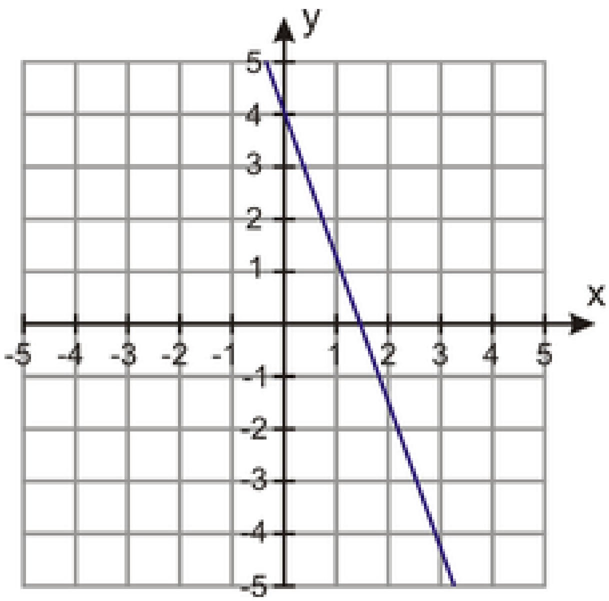
Now we want to apply this process in reverse—to start with the graph of the line and write the equation of the line in standard form.

Example 3

Find the equation of each line and write it in standard form.

a) 

b) 

c) 

Solution

a) We see that the $x-$intercept is $(3, 0) \Rightarrow x = 3$ and the $y-$intercept is $(0, -4) \Rightarrow y = -4$

We saw that in standard form $ax + by = c$: if we "cover up" the y term, we get $ax = c$, and if we "cover up" the x term, we get $by = c$.

So we need to find values for a and b so that we can plug in 3 for x and -4 for y and get the same value for c in both cases. This is like finding the least common multiple of the $x-$ and $y-$intercepts.

In this case, we see that multiplying $x = 3$ by 4 and multiplying $y = -4$ by –3 gives the same result:

$$(x = 3) \times 4 \Rightarrow 4x = 12 \quad \text{and} \quad (y = -4) \times (-3) \Rightarrow -3y = 12$$

Therefore, $a = 4, b = -3$ and $c = 12$ and **the equation in standard form is** . $4x - 3y = 12$

b) We see that the $x-$intercept is $(3, 0) \Rightarrow x = 3$ and the $y-$intercept is $(0, 3) \Rightarrow y = 3$

The values of the intercept equations are already the same, so $a = 1, b = 1$ and $c = 3$. **The equation in standard form is** . $x + y = 3$

c) We see that the $x-$intercept is $\left(\frac{3}{2}, 0\right) \Rightarrow x = \frac{3}{2}$ and the $y-$intercept is $(0, 4) \Rightarrow y = 4$

Let's multiply the $x-$intercept equation by $2 \Rightarrow 2x = 3$

Then we see we can multiply the $x-$intercept again by 4 and the $y-$intercept by 3, so we end up with $8x = 12$ and $3y = 12$.

The equation in standard form is $8x + 3y = 12$.